Licenciaturas em Engenharia de Energias Renováveis
Matemática I - 2011/2012

Exame Época Normal - 8 de Fevereiro 2012

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- Apenas é permitido consultar o formulário.
- Não é permitido utilizar máquina de calcular.
- Explicite detalhadamente todos os cálculos efectuados.

Duração: 2h00

1. Considere o sistema de equações lineares $\begin{cases} x+2y=-1 \\ 3x+4y=-1 \end{cases}$
   (a) Escreva o sistema na forma matricial $Ax=b$.
   (b) Mostre que este sistema tem solução única.
   (c) Calcule a inversa da matriz dos coeficientes do sistema.
   (d) Utilize a matriz inversa, calculada na alínea anterior, para resolver o sistema.

2. Considere a matriz $A=\begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}$
   (a) Calcule o traço de A.
   (b) Calcule os valores próprios de A.
   (c) Determine um vector próprios associado a cada um dos valores próprios.

3. Determine a equação da recta tangente à função $f(x)$ em $x=1$, sabendo que $\int f(x)\,dx = x^3+2/x+2$.

4. Calcule as seguintes primitivas
   (a) $\int \cos\left(\frac{x}{\sqrt{2}}\right)\,dx$
   (b) $\int \frac{x^4+2x-3}{x^2}\,dx$
   (c) $\int e^x(2x+3)\,dx$
   (d) $\int x\sin(2x)\,dx$
   (e) $\int x\sqrt{x^2-1}\,dx$

5. Considere as funções $f(x)=-x^3$ e $g(x)=-x$
   (a) Faça os gráficos destas duas funções, indicando os pontos de intersecção.
   (b) Calcule a área compreendida entre os dois gráficos.
   (c) Calcule o volume definido pela rotação do gráfico de $f(x)$, em torno do eixo x, desde $x=0$ até $x=1$.

1)

$$\begin{cases} x + 2y = -1 \\ 3x + 4y = -1 \end{cases}$$

a) $\begin{cases} x + 2y = -1 \\ 3x + 4y = -1 \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \Leftrightarrow Ax = b$

b) $[A \mid I] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \underset{\Leftrightarrow}{l_3 \leftarrow l_3 - 3l_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \underset{\Leftrightarrow}{l_2 \leftarrow -\frac{1}{2} l_2}$

$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right] \underset{\Leftrightarrow}{l_1 \leftarrow l_1 - 2 l_2} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right] = [I \mid A^{-1}]$

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

c) $Ax = b \Leftrightarrow A^{-1}A x = A^{-1} b \Leftrightarrow I x = A^{-1} b \Leftrightarrow x = A^{-1} b$

$$x = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} (-2)(-1) + (1)(-1) \\ (3/2)(-1) + (-1/2)(-1) \end{bmatrix} = \begin{bmatrix} 2 - 1 \\ -3/2 + 1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2/2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} x = 1 \\ y = -1 \end{cases}$$

2) $A = \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}$

a) $Tr(A) = 1 + (-2) = 1 - 2 = \underline{\underline{-1}} \left(= \lambda_1 + \lambda_2 \right)$

b) $|A - \lambda I| = 0 \Leftrightarrow \begin{vmatrix} 1-\lambda & 4 \\ 1 & -2-\lambda \end{vmatrix} = 0 \Leftrightarrow (1-\lambda)(-2-\lambda) - (1)(4) = 0$

$\Leftrightarrow -2 - \lambda + 2\lambda + \lambda^2 - 4 = 0 \Leftrightarrow \lambda^2 + \lambda - 6 = 0$

$\Leftrightarrow \lambda = \dfrac{-1 \pm \sqrt{1^2 - (4)(1)(-6)}}{(2)(1)} = \dfrac{-1 \pm \sqrt{1+24}}{2}$

$\Leftrightarrow \lambda = \dfrac{-1 \pm \sqrt{25}}{2} = \dfrac{-1 \pm 5}{2} \Leftrightarrow \lambda = \dfrac{-1-5}{2} \vee \lambda = \dfrac{-1+5}{2}$

$\Leftrightarrow \lambda = -3 \vee \lambda = 2$

$\underline{\lambda_1 = -3 \quad e \quad \lambda_2 = 2}$

c) $\lambda = \lambda_1 = -3$

$(A - \lambda_1 I) X_1 = 0 \Leftrightarrow \begin{bmatrix} 1+3 & 4 \\ 1 & -2+3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Leftrightarrow \left[\begin{array}{cc|c} 4 & 4 & 0 \\ 1 & 1 & 0 \end{array}\right] \underset{\Leftrightarrow}{l_1 \leftrightarrow l_2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 4 & 4 & 0 \end{array}\right] \underset{\Leftrightarrow}{l_2 \leftarrow l_2 - 4l_1} \left[\begin{array}{cc|c} 4 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{cases} 4x + 4y = 0 \\ 0 = 0 \end{cases}$

$\Leftrightarrow \begin{cases} 4x = -4y \\ 0 = 0 \end{cases} \Leftrightarrow \begin{cases} x = -y \\ y = \alpha \in \mathbb{R} \setminus \{0\} \end{cases} \Rightarrow X_1 = \begin{bmatrix} -\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, se $\alpha = 1 \Rightarrow X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = \lambda_2 = 2$

$(A - \lambda_2 I) X_2 = 0 \Leftrightarrow \begin{bmatrix} 1-2 & 4 \\ 1 & -2-2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\underset{\Leftrightarrow}{l_2 \leftarrow l_2 + l_1} \left[\begin{array}{cc|c} -1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{cases} -x + 4y = 0 \\ 0 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 4y \\ 0 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 4y \\ y = \alpha \in \mathbb{R} \setminus \{0\} \end{cases}$

$X_2 = \begin{bmatrix} 4\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, se $\alpha = 1$ obtemos $X_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

3)

$$\int f(x)\,dx = x^3 + \frac{2}{x} + 2$$

$$f(x) = \left(x^3 + \frac{2}{x} + 2\right)' = (x^3)' + \left(\frac{2}{x}\right)' + (2)' = 3x^2 + \frac{(2)'(x) - (2)(x)'}{x^2} + 0$$

$$= 3x^2 + \frac{0-2}{x^2} = 3x^2 - \frac{2}{x^2}$$

$$\boxed{y = mx + b}$$

$$f'(x) = \left(3x^2 - \frac{2}{x^2}\right)' = 6x - \frac{(2)'(x^2) - (2)(x^2)'}{x^4} = 6x - \frac{0 - 4x}{x^4}$$

$$= 6x + \frac{4x}{x^4} = 6x + \frac{4}{x^3}$$

$$m = f'(1) = (6)(1) + \frac{4}{(1)^3} = 6 + 4 = 10$$

$$\boxed{y = 10x + b}$$

se $x = 1 \Rightarrow y = ?$

$$f(1) = 3 \times 1^2 - \frac{2}{1^2} = 3 - 2 = 1$$

$$1 = 10(1) + b \Leftrightarrow b = 1 - 10 = -9$$

$$\boxed{y = 10x - 9}$$

4) a) $\int \cos\left(\frac{x}{\sqrt{2}}\right) dx$

$$u = \frac{x}{\sqrt{2}}$$

$$\frac{du}{dx} = \left(\frac{x}{\sqrt{2}}\right)' = \frac{1}{\sqrt{2}} \Leftrightarrow du = \frac{1}{\sqrt{2}} dx$$

$$\int \cos\left(\frac{x}{\sqrt{2}}\right) dx = \sqrt{2} \int \cos\left(\frac{x}{\sqrt{2}}\right) \cdot \frac{1}{\sqrt{2}} \cdot dx = \sqrt{2} \int \cos(u)\, du$$

$$= \sqrt{2}\, \text{sen}(u) + C = \boxed{\sqrt{2} \cdot \text{sen}\left(\frac{x}{\sqrt{2}}\right) + C}$$

b) $\int \frac{x^4 + 2x - 3}{x^2} dx = \int \frac{x^4}{x^2} + \frac{2x}{x^2} - \frac{3}{x^2} dx = \int x^2 + \frac{2}{x} - \frac{3}{x^2} dx$

$$= \int x^2 dx + 2 \int \frac{1}{x} dx - 3 \int x^{-2} dx = \frac{x^3}{3} + 2 \ln|x| - 3 \frac{x^{-1}}{-1} + C$$

$$= \boxed{\frac{x^3}{3} + 2 \ln|x| + \frac{1}{x} + C}$$

c) $\int e^x (2x+3)\, dx = e^x (2x+3) - \int e^x (2x+3)'\, dx = e^x (2x+3) - 2 \int e^x dx$

$$= \boxed{e^x (2x+3) - 2e^x + C}$$

d) $\int x \sin(2x)\, dx = -\frac{\cos(2x)}{2} * x - \int -\frac{\cos(2x)}{2} (x)' dx = -\frac{\cos(2x)}{2} x - \frac{1}{2} \int -\cos(2x)\, dx$

$$= -\frac{x}{2} \cos(2x) + \frac{1}{2} * \frac{\text{sen}(2x)}{2} + C = \boxed{-\frac{x}{2} \cos(2x) + \frac{1}{4} \text{sen}(2x) + C}$$

e) $\int x \sqrt{x^2 - 1}\, dx$

$$u = x^2 - 1 \; ; \; \frac{du}{dx} = 2x \Leftrightarrow du = 2x\, dx$$

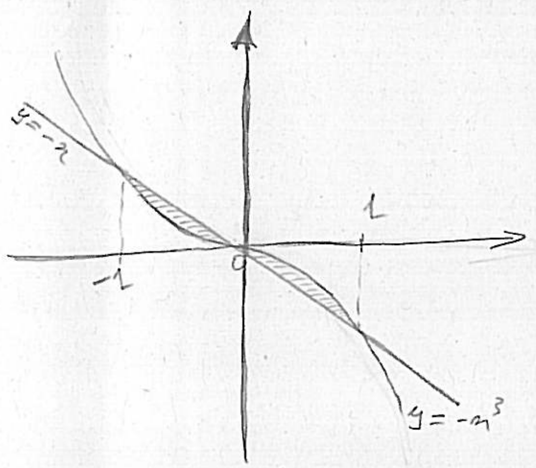
$$= \frac{1}{2} \int \sqrt{x^2 - 1} \cdot 2x\, dx = \frac{1}{2} \int \sqrt{u}\, du = \frac{1}{2} \int u^{\frac{1}{2}} dx = \frac{1}{2} \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{1}{2} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} = \frac{1}{2} \times \frac{2}{3} \times u^{\frac{3}{2}} + C = \frac{1}{3} (\sqrt{u})^3 + C$$

$$= \frac{\sqrt{u}^3}{3} + C = \boxed{\frac{\sqrt{x^2-1}}{3} + C}$$

5) $f(x) = -x^3$ e $g(x) = -x$

a)



$-x^3 = -x \Leftrightarrow x^3 = x \Leftrightarrow x^3 - x = 0$

$\Leftrightarrow x(x^2-1) = 0 \Leftrightarrow x = 0 \vee x^2 - 1 = 0$

$\Leftrightarrow x = 0 \vee x^2 = 1 \Leftrightarrow x = 0 \vee x^2 = \pm\sqrt{1}$

$\Leftrightarrow x = 0 \vee x = -1 \vee x = 1$

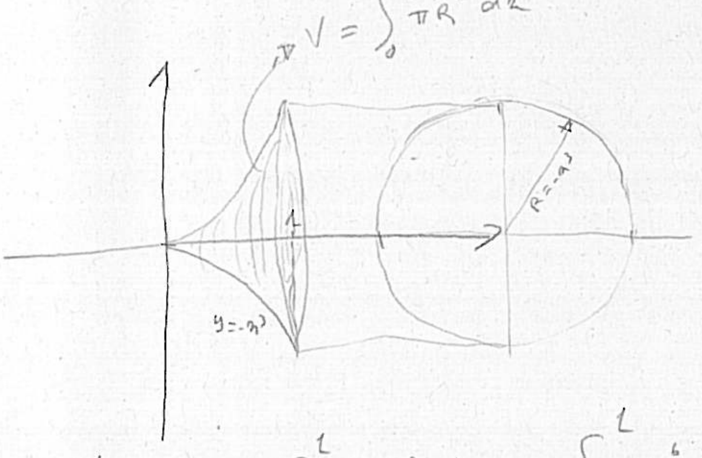
b) $\text{Área} = \int_{-1}^{0} (-x) - (-x^3)\, dx + \int_{0}^{1} (-x^3) - (-x)\, dx$

$= \int_{-1}^{0} -x + x^3\, dx + \int_{0}^{1} -x^3 + x\, dx = \left[-\frac{x^2}{2} + \frac{x^4}{4}\right]_{-1}^{0} + \left[-\frac{x^4}{4} + \frac{x^2}{2}\right]_{0}^{1}$

$= 0 - \left(-\frac{(-1)^2}{2} + \frac{(-1)^4}{4}\right) + \left(-\frac{1^4}{4} + \frac{1^2}{2}\right) - 0 = \frac{1}{2} - \frac{1}{4} - \frac{1}{4} + \frac{1}{2} = \frac{2}{2} - \frac{2}{4}$

$= 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$

c)



$V = \int_0^1 \pi R^2\, dx = \int_0^1 \pi(-x^3)^2\, dx = \int_0^1 \pi x^6\, dx = \pi \int_0^1 x^6\, dx = \pi \left[\frac{x^7}{7}\right]_0^1$

$= \pi \left(\frac{1^7}{7} - \frac{0^7}{7}\right) = \boxed{\frac{\pi}{7}}$