Licenciaturas em Engenharia de Energias Renováveis
Matemática I - 2011/2012

Exame Época Normal - 17 de Janeiro 2012

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- Apenas é permitido consultar o formulário.
- Não é permitido utilizar máquina de calcular.
- Explicite detalhadamente todos os cálculos efectuados.

Duração: 2h00

1. Considere o sistema de equações lineares $\begin{cases} 2x - y + z = 2 \\ 3x - z = 2 \\ x + 2y + 3z = 6 \end{cases}$

   (a) Mostre que a solução deste sistema é única (sem a calcular).

   (b) Calcule a solução pelo método de Cramer.

2. Considere a matriz $A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix}$

   (a) Calcule todos os valores próprios de A.

   (b) Determine um vector próprio associado a cada um dos dois valores próprios de menor maganitude (se não resolveu a alínea anterior considere $\lambda_1 = -1$ e $\lambda_2 = 2$).

3. Determine a equação da recta tangente à circunferência de equação $y^2 + x^2 = 4$ no ponto $(1;\ \sqrt{3})$.

4. Calcule os seguintes integrais definidos

   (a) $\int_0^\pi x\cos(x)\,dx$

   (b) $\int_1^2 \frac{1}{2x^2} - 5x^3 + \sqrt{x}\,dx$

   (c) $\int_0^1 \frac{2x+3}{(x^2+3x-7)^2}\,dx$

   (d) $\int_{-1}^0 \frac{1}{4-5x}\,dx$

5. Considere as funções $f(x) = 4 - x^2$ e $g(x) = x + 2$

   (a) Faça os gráficos destas duas funções, indicando os pontos de intersecção

   (b) Calcule a área compreendida entre os dois gráficos

   (c) Calcule o volume definido pela rotação do gráfico de $f(x)$, desde $x = -2$ até $x = 2$, em torno do eixo x

1)

$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix} \Leftrightarrow Ax = b$$

a)

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 3 & 0 & -1 \\ 1 & 2 & 3 \end{vmatrix} \begin{matrix} \\ \\ \\ \end{matrix} = 0 + 6 + 1 - 0 + 4 + 9 = 20$$

$$\begin{matrix} 2 & -1 & 1 \\ 3 & 0 & -1 \end{matrix}$$

$|A| = 20 \neq 0 \Rightarrow$ Sistema com solução única

b)

$$x = \frac{|A_1|}{|A|}, \quad y = \frac{|A_2|}{|A|}, \quad z = \frac{|A_3|}{|A|}$$

$$|A_1| = \begin{vmatrix} 2 & -1 & 1 \\ 2 & 0 & -1 \\ 6 & 2 & 3 \end{vmatrix} = 0 + 4 + 6 - 0 + 4 + 6 = 20$$

$$\begin{matrix} 2 & -1 & 1 \\ 2 & 0 & -1 \end{matrix}$$

$$|A_2| = \begin{vmatrix} 2 & 2 & 1 \\ 3 & 2 & -1 \\ 1 & 6 & 3 \end{vmatrix} = 12 + \cancel{18} - 2 - 2 + 12 - \cancel{18} = 20$$

$$\begin{matrix} 2 & 2 & 1 \\ 3 & 2 & -1 \end{matrix}$$

$$|A_3| = \begin{vmatrix} 2 & -1 & 2 \\ 3 & 0 & 2 \\ 1 & 2 & 6 \end{vmatrix} = 0 + 12 - 2 + 0 - 8 + 18 = 20$$

$$\begin{matrix} 2 & -1 & 2 \\ 3 & 0 & 2 \end{matrix}$$

$$x = \frac{|A_1|}{|A|} = \frac{20}{20} = 1, \quad y = \frac{|A_2|}{|A|} = \frac{20}{20} = 1 \text{ e } z = \frac{|A_3|}{|A|} = \frac{20}{20} = 1$$

[2]

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

a)

$$|A - \lambda I| = 0 \Leftrightarrow \begin{vmatrix} 2-\lambda & 2 & 3 \\ 1 & 2-\lambda & 1 \\ 2 & -2 & 1-\lambda \end{vmatrix} = 0 \Leftrightarrow (2-\lambda)(2-\lambda)(1-\lambda) - 6 + 4 - 6(2-\lambda)$$

$$\begin{array}{ccc} 2-\lambda & 2 & 3 \\ 1 & 2-\lambda & 1 \end{array}$$

$$+ 2(2-\lambda) - 2(1-\lambda) = 0$$

$$\Leftrightarrow (2-\lambda)^2(1-\lambda) - 2 - 6(2-\lambda) + 2(2-\lambda) - 2(1-\lambda) = 0$$

$$\Leftrightarrow (2-\lambda)^2(1-\lambda) - 2 - 12 + 6\lambda + 4 - 2\lambda - 2 + 2\lambda = 0$$

$$\Leftrightarrow (2-\lambda)^2(1-\lambda) - 12 + 6\lambda = 0$$

$$\Leftrightarrow (2-\lambda)^2(1-\lambda) - 6(2-\lambda) = 0$$

$$\Leftrightarrow (2-\lambda)\left[(2-\lambda)(1-\lambda) - 6\right] = 0$$

$$\Leftrightarrow (2-\lambda)(2 - 2\lambda - \lambda + \lambda^2 - 6) = 0$$

$$\Leftrightarrow (2-\lambda)(\lambda^2 - 3\lambda - 4) = 0$$

$$\Leftrightarrow 2 - \lambda = 0 \quad \vee \quad \lambda^2 - 3\lambda - 4 = 0$$

$$\Leftrightarrow \lambda = 2 \quad \vee \quad \lambda = \frac{3 \pm \sqrt{9+16}}{2}$$

$$\Leftrightarrow \lambda = 2 \quad \vee \quad \lambda = \frac{3 \pm 5}{2}$$

$$\Leftrightarrow \lambda = 2 \quad \vee \quad \lambda = -1 \quad \vee \quad \lambda = 4$$

Valores próprios de A: $\lambda_1 = -1$, $\lambda_2 = 2$ e $\lambda_3 = 4$

b)

$$(A-\lambda_1 I)X_1 = 0 \Leftrightarrow \begin{bmatrix} 2-(-1) & 2 & 3 \\ 1 & 2-(-1) & 1 \\ 2 & -2 & 1-(-1) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 3 & 2 & 3 \\ 1 & 3 & 1 \\ 2 & -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \underset{\Leftrightarrow}{\ell_1 \leftrightarrow \ell_2} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 3 & 2 & 3 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right] \begin{array}{l} \ell_2 \leftarrow \ell_2 - 3\ell_1 \\ \ell_3 \leftarrow \ell_3 - 2\ell_1 \\ \Leftrightarrow \end{array}$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & -8 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{cases} x+3y+z=0 \\ -7y=0 \\ -8y=0 \end{cases} \Leftrightarrow \begin{cases} x=-z \\ y=0 \\ y=0 \end{cases} \Leftrightarrow \begin{cases} x=-z \\ y=0 \\ z \in \mathbb{R}\setminus\{0\} \end{cases}$$

$z = t \in \mathbb{R}\setminus\{0\}$

$$X_1 = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ ; se } t=1 \text{ obtemos } X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$(A-\lambda_2 I)X_2 = 0 \Leftrightarrow \begin{bmatrix} 2-2 & 2 & 3 \\ 1 & 2-2 & 1 \\ 2 & -2 & 1-2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 0 & 2 & 3 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & -2 & -1 & 0 \end{array}\right] \underset{\Leftrightarrow}{\ell_2 \leftrightarrow \ell_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 2 & -2 & -1 & 0 \end{array}\right] \begin{array}{l} \ell_3 \leftarrow \ell_3 - 2\ell_1 \\ \Leftrightarrow \end{array}$$

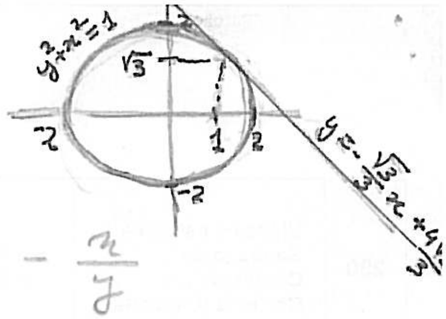
$$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & -2 & -3 & 0 \end{array}\right] \underset{\Leftrightarrow}{\ell_3 \leftarrow \ell_3 + \ell_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{cases} x+z=0 \\ 2y+3z=0 \\ 0=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x=-z \\ y=-\frac{3}{2}z \\ z \in \mathbb{R}\setminus\{0\} \end{cases} \qquad z = t \in \mathbb{R}\setminus\{0\}$$

$$X_2 = \begin{bmatrix} -t \\ -\frac{3}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -\frac{3}{2} \\ 1 \end{bmatrix} \text{ ; se } t=1 \text{ obtemos } X_2 = \begin{bmatrix} -1 \\ -3/2 \\ 1 \end{bmatrix}$$

3

$$(y^2+x^2)' = (4)' \Leftrightarrow 2y\cdot y' + 2x = 0$$
$$\Leftrightarrow y' = -\frac{2x}{2y} \Leftrightarrow y' = -\frac{x}{y}$$



$$y^2 + 1 = 4 \Leftrightarrow y^2 = 3 \Leftrightarrow y = \sqrt{3}$$

$$y'(1,\sqrt{3}) = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$$m = \frac{y-y_0}{x-x_0} \Leftrightarrow m(x-x_0) = y-y_0 \Leftrightarrow y = m(x-x_0)+y_0$$

$$y = -\frac{\sqrt{3}}{3}(x-1)+\sqrt{3} \Leftrightarrow y = -\frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{3} + \sqrt{3}$$

$$\Leftrightarrow \boxed{y = -\frac{\sqrt{3}}{3}x + \frac{4\sqrt{3}}{3}}$$

<u>Verificação</u>: Se $x = 1 \Rightarrow y = -\frac{\sqrt{3}}{3} + \frac{4\sqrt{3}}{3} = \frac{3\sqrt{3}}{3} = \sqrt{3}$ ✓

4 a)

$$\int x\cos(x)\,dx = \text{sen}(x)\,x - \int \text{sen}(x)\,dx$$

$$= \text{Sen}(x)\,x - (-\cos(x))$$

$$= \text{Sen}(x)\,x + \cos(x)$$

Verificação:

$$(\text{Sen}(x)\,x + \cos(x))' = \cos(x)\,x + \text{Sen}(x) - \text{sen}(x) = \cos(x)(x)$$

$$= x\cos(x)$$

$$\int_0^{\pi} x\cos(x)\,dx = \left[x\,\text{Sen}(x) + \cos(x)\right]_0^{\pi} = \pi\,\text{sen}(\pi) + \cos(\pi) - 0 - \cos(0)$$

$$= \pi * 0 + (-1) - 1 = -2$$

— 4 —

b) $\int \frac{1}{2x^2} - 5x^3 + \sqrt{x}\,dx$

$$\frac{1}{2}\int x^{-2}\,dx - 5\int x^3\,dx + \int x^{\frac{1}{2}} = \frac{1}{2}\frac{x^{-1}}{(-1)} - 5\frac{x^4}{4} + \frac{x^{3/2}}{3/2}$$

$$= -\frac{1}{2x} - \frac{5}{4}x^4 + \frac{2}{3}\sqrt{x}^3$$

$$\int_1^2 \frac{1}{2x^2} - 5x^3 + \sqrt{x}\,dx = \left[\frac{-1}{2x} - \frac{5}{4}x^4 + \frac{2}{3}\sqrt{x}^3\right]_1^2 =$$

$$= \left(-\frac{1}{2} - \frac{5}{4}(2)^4 + \frac{2}{3}\sqrt{2}^3\right) - \left(\frac{1}{2} - \frac{5}{4}(1)^4 + \frac{2}{3}\sqrt{1}^3\right)$$

$$= \left(-\frac{1}{2} - \frac{5}{4}\times 16 + \frac{2}{3}\sqrt{8}\right) - \frac{1}{2} + \frac{5}{4} - \frac{2}{3} = -1 - 20 + \frac{4}{3}\sqrt{2} + \frac{5}{4} - \frac{2}{3}$$

$$= -21 + \frac{4}{3}\sqrt{2} + \frac{15-8}{12} = -21 + \frac{4\sqrt{2}}{3} + \frac{7}{12}$$

$$= \frac{-252 + 16\sqrt{2} + 7}{12} = \frac{16\sqrt{2} - 245}{12}$$

c) $\int_0^1 \frac{2x+3}{(x^2+3x-7)^2}\,dx$

$u = x^2+3x-7$
$du = (2x+3)\,dx$
$u(0) = -7$
$u(1) = 1^2+3(1)-7 = -3$

$= \int_{-7}^{-3} \frac{1}{u^2}\,du = \int_{-7}^{-3} u^{-2}\,du$

$= \left[\frac{u^{-1}}{-1}\right]_{-7}^{-3} = \left[-\frac{1}{u}\right]_{-7}^{-3} = \left(-\frac{1}{-3}\right) - \left(-\frac{1}{-7}\right) = \frac{1}{3} - \frac{1}{7} =$

$= \frac{7-3}{21} = \frac{4}{21}$

d)

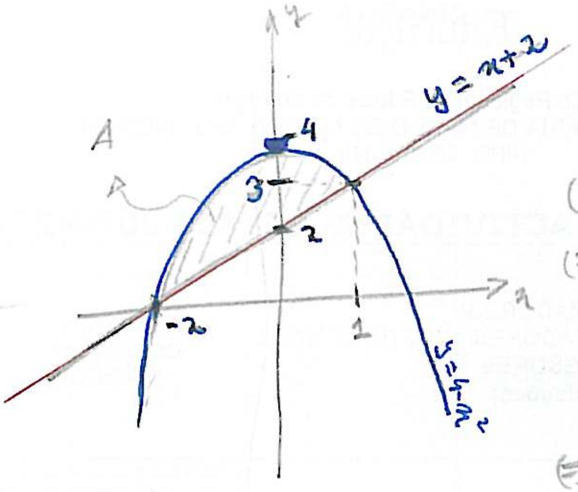
$\int_{-1}^{0} \frac{1}{4-5x}\,dx = -\frac{1}{5}\int_{-1}^{0} \frac{-5}{4-5x}\,dx = -\frac{1}{5}\left[\ln|4-5x|\right]_{-1}^{0}$

$= -\frac{1}{5}\left[\ln|4| - \ln|4+5|\right] = -\frac{1}{5}\left(\ln(4) - \ln(9)\right)$

$= -\frac{1}{5}\ln(2^2) + \frac{1}{5}\ln(3^2) = \frac{2}{5}\ln(3) - \frac{2}{5}\ln(2)$

5

a) $f(x) = 4 - x^2$ e $g(x) = x + 2$



$4 - x^2 = x + 2$

$\Leftrightarrow -x^2 - x + 2 = 0$

$\Leftrightarrow x^2 + x - 2 = 0$

$x = \dfrac{-(1) \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2}$

$\Leftrightarrow x = \dfrac{-1 \pm \sqrt{1+8}}{2} = \dfrac{-1 \pm \sqrt{9}}{2}$

$\Leftrightarrow x = \dfrac{-1-3}{2} \vee x = \dfrac{-1+3}{2}$

$\Leftrightarrow x = -2 \vee x = 1$

b) Área $= A = \displaystyle\int_{-2}^{1} (4 - x^2) - (x+2)\, dx = \int_{-2}^{1} 4 - x^2 - x - 2\, dx = \int_{-2}^{1} -x^2 - x + 2\, dx$

$= \left[-\dfrac{x^3}{3} - \dfrac{x^2}{2} + 2x \right]_{-2}^{1} = \left(-\dfrac{1}{3} - \dfrac{1}{2} + 2 \right) - \left(-\dfrac{(-8)}{3} - \dfrac{4}{2} - 4 \right)$

$= -\dfrac{1}{3} - \dfrac{1}{2} + 2 - \dfrac{8}{3} + \dfrac{4}{2} + 4 = -\dfrac{9}{3} + \dfrac{3}{2} + 6 = 3 + \dfrac{3}{2}$

$= \dfrac{6+3}{2} = 9/2$

c)

$$V = \int_{-2}^{2} \pi (4-x^2)^2 \, dx = \pi \int_{-2}^{2} (16 - 8x^2 + x^4) \, dx$$

$$= \pi \left[16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_{-2}^{2} = \pi \left(16 \times 2 - \frac{8 \times 8}{3} + \frac{32}{5} \right) - \pi \left(-16 \times 2 + \frac{8 \times 8}{3} + \frac{32}{5} \right)$$

$$= \pi \left(32 - \frac{64}{3} + \frac{32}{5} \right) - \pi \left(-32 + \frac{64}{3} + \frac{32}{5} \right)$$

$$= \pi \left(32 - \frac{64}{3} + \frac{32}{5} + 32 - \frac{64}{3} - \frac{32}{5} \right)$$

$$= \pi \left(64 - 2 \times \frac{64}{3} \right) = \pi \left(\frac{192 - 128}{3} \right)$$

$$= \pi \left(\frac{64}{3} \right) = \frac{64}{3} \pi$$