Exercício 1 : $u = \begin{bmatrix} 2 \\ a \\ 3 \end{bmatrix}$; $v = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$; $x = \begin{bmatrix} -4 \\ -4 \\ -2 \end{bmatrix}$; $y = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

a) $\|v\| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{1+4+1} = \sqrt{6}$

$\|y\| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$

b) $u \perp v$ se $u^T . v = 0 \Leftrightarrow \begin{bmatrix} 2 & a & 3 \end{bmatrix} . \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 0$

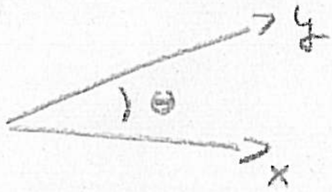
$\Leftrightarrow (2)(1) + (a)(-2) + (3)(1) = 0$

$\Leftrightarrow 2 - 2a + 3 = 0$

$\Leftrightarrow -2a = -5$

$\Leftrightarrow a = \frac{5}{2}$

c) $x \parallel y$ se $\cos(\theta) = \pm 1$ com θ o ângulo entre x e y :



$$\cos(\theta) = \frac{x^T . y}{\|x\| . \|y\|} = \frac{(-4)(2) + (-4)(2) + (-2)(-1)}{\sqrt{(-4)^2 + (-4)^2 + 2^2} \sqrt{(2)^2 + (2)^2 + (-1)^2}}$$

$$= \frac{-8 - 8 - 2}{\sqrt{16+16+4} . \sqrt{4+4+1}} = \frac{-18}{\sqrt{36} . \sqrt{9}} = \frac{-18}{(6)(3)} = \frac{-18}{18} = -1$$

$\cos(\theta) = -1 \Rightarrow x$ e y são colineares.

d) $Proj_{v}(x) = C_{v}(x) \cdot \frac{v}{\|v\|} = \frac{x^T \cdot v}{\|v\|} \cdot \frac{v}{\|v\|}$

$$= \frac{x^T \cdot v}{\|v\|^2} \cdot v$$

$$= \left(\frac{(-4)(1) + (-4)(-2) + (2)(1)}{(1)^2 + (-2)^2 + (1)^2}\right) \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$= \left(\frac{-4 + 8 + 2}{1 + 4 + 1}\right) \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$= \left(\frac{6}{6}\right) \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$= (1) \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = v$$

$\Leftrightarrow Proj_{v}(x) = v$

Exercício 2: $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 2 & -1 \\ -4 & 4 & -3 \end{bmatrix}$; $B = \begin{bmatrix} -2 & 1 \\ 1 & 0 \\ -2 & -1 \end{bmatrix}$ e $C = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix}$

a) $A + B$ não faz sentido pois não têm as mesmas dimensões.
$B + C$ não faz sentido

$$B^T - C = \begin{bmatrix} -2 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -2-1 & 1-0 & -2+(-1) \\ 1-3 & 0-1 & -1-0 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 1 & -1 \\ -2 & -1 & -1 \end{bmatrix}$$

b)
$$A^2 = A.A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 2 & -1 \\ -4 & 4 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 1 \\ -1 & 2 & -1 \\ -4 & 4 & -3 \end{bmatrix} = \begin{bmatrix} 1+2-4 & -2-4+4 & 1+2-3 \\ -1-2+4 & 2+4-4 & -1-2+3 \\ -4-4+12 & 8+8-12 & -4-4+9 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 & 0 \\ 1 & 2 & 0 \\ 4 & 4 & 1 \end{bmatrix}$$

$A.C$: Não faz sentido pois o nº de colunas de A (3) é diferente do nº de linhas de C (2).

$$B.C = \begin{bmatrix} -2 & 1 \\ 1 & 0 \\ -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -2+3 & 0+1 & 2+0 \\ 1+0 & 0+0 & -1+0 \\ -2-3 & 0-1 & 4+0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ -5 & -1 & 2 \end{bmatrix}$$

$$C.B = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ 1 & 0 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -2+0+2 & 1+0+1 \\ -6+1+0 & 3+0+0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -5 & 3 \end{bmatrix}$$

c) $(C.B)^{-1} = \begin{bmatrix} 0 & 2 \\ -5 & 3 \end{bmatrix}^{-1}$

Vamos calcular a inversa por um dos métodos estudados

- Método da matriz adjunta
- ou método de Gauss-Jordan

Pelo método de Gauss-Jordan

$$\left[\begin{array}{cc|cc} 0 & 2 & 1 & 0 \\ -5 & 3 & 0 & 1 \end{array}\right]_{L_1 \leftrightarrow L_2} \Leftrightarrow \left[\begin{array}{cc|cc} -5 & 3 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{array}\right]_{L_2 \leftarrow \frac{1}{2}L_2}$$

$$\Leftrightarrow \left[\begin{array}{cc|cc} -5 & 3 & 0 & 1 \\ 0 & 1 & \frac{1}{2} & 0 \end{array}\right]_{L_1 \leftarrow L_1 - 3L_2} \Leftrightarrow \left[\begin{array}{cc|cc} -5 & 0 & -\frac{3}{2} & 1 \\ 0 & 1 & \frac{1}{2} & 0 \end{array}\right]_{L_1 \leftarrow -\frac{1}{5}L_1}$$

$$\Leftrightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{10} & -\frac{1}{5} \\ 0 & 1 & \frac{1}{2} & 0 \end{array}\right]$$

$$(C.B)^{-1} = \begin{bmatrix} \frac{3}{10} & -\frac{1}{5} \\ \frac{1}{2} & 0 \end{bmatrix}$$

· Exercício 3:

$$\begin{cases} 2x - y + az = 8 \\ x + 2y + 3z = 9 \\ 3x - z = 3 \end{cases} \Leftrightarrow \begin{bmatrix} 2 & -1 & a \\ 1 & 2 & 3 \\ 3 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ 3 \end{bmatrix} \Leftrightarrow Ax = b$$

a) O sistema tem solução única se $|A| \neq 0$

$$|A| = \left|\begin{array}{ccc|cc} 2 & -1 & a & 2 & -1 \\ 1 & 2 & 3 & 1 & 2 \\ 3 & 0 & -1 & 3 & 0 \end{array}\right| = -4 - 9 + 0 - 6a - 0 - 1 \neq 0$$

$$\Leftrightarrow -14 - 6a \neq 0 \Leftrightarrow a \neq -\frac{14}{6} \Leftrightarrow a \neq -\frac{7}{3}$$

b) Se $a = 1 \Rightarrow |A| = -14 - 6(1) = -20$

$$|A_1| = \left|\begin{array}{ccc|cc} 8 & -1 & 1 & 8 & -1 \\ 9 & 2 & 3 & 9 & 2 \\ 3 & 0 & -1 & 3 & 0 \end{array}\right| = -16 - 9 + 0 - 6 - 0 - 9 = -40$$

$$|A_2| = \left|\begin{array}{ccc|cc} 2 & 8 & 1 & 2 & 8 \\ 1 & 9 & 3 & 1 & 9 \\ 3 & 3 & -1 & 3 & 3 \end{array}\right| = -18 + 72 + 3 - 27 - 18 + 8 = 20$$

$$|A_3| = \left|\begin{array}{ccc|cc} 2 & -1 & 8 & 2 & -1 \\ 1 & 2 & 9 & 1 & 2 \\ 3 & 0 & 3 & 3 & 0 \end{array}\right| = 12 - 27 + 0 - 48 - 0 + 3 = -60$$

$$\begin{cases} x = \frac{|A_1|}{|A|} \\ y = \frac{|A_2|}{|A|} \\ z = \frac{|A_3|}{|A|} \end{cases} \Leftrightarrow \begin{cases} x = \frac{-40}{-20} \\ y = \frac{20}{-20} \\ z = \frac{-60}{-20} \end{cases} \Leftrightarrow \begin{cases} x = 2 \\ y = -1 \\ z = 3 \end{cases}$$

Exercício 4 $A = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$

a) A não é simétrica pois $A \neq A^T$:

$$\begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} \neq \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$$

b) $Tra(A) = 0 + 3 = 3$

c) $|A - \lambda I| = 0 \Leftrightarrow \begin{vmatrix} 0-\lambda & -1 \\ 2 & 3-\lambda \end{vmatrix} = 0 \Leftrightarrow (-\lambda)(3-\lambda) - (2)(-1) = 0$

$\Leftrightarrow -3\lambda + \lambda^2 + 2 = 0 \Leftrightarrow \lambda^2 - 3\lambda + 2 = 0$

$\Leftrightarrow \lambda = \dfrac{3 \pm \sqrt{(-3)^2 - (4)(2)}}{2} \Leftrightarrow \lambda = \dfrac{3 \pm \sqrt{9-8}}{2} \Leftrightarrow \lambda = \dfrac{3 \pm 1}{2}$

$\Leftrightarrow \lambda = 1 \vee \lambda = 2$

$\lambda(A) = \{\lambda_1; \lambda_2\} = \{1; 2\}$

d) $(A - \lambda_1 I) X_1 = 0 \Leftrightarrow \begin{bmatrix} 0-\lambda_1 & -1 \\ 2 & 3-\lambda_1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow$ com $\lambda_1 = 1$

$\Leftrightarrow \begin{bmatrix} -1 & -1 \\ 2 & 3-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Leftrightarrow \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 2 & 2 & 0 \end{array}\right] \underset{L_2 \leftarrow L_2 + 2L_1}{\Leftrightarrow} \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{cases} -x - y = 0 \\ 0 = 0 \end{cases} \Leftrightarrow \begin{cases} x = -y \\ y \in \mathbb{R} \setminus \{0\} \end{cases}$

$\Leftrightarrow \begin{cases} x = -\alpha \\ y = \alpha \end{cases}$ com $\alpha \in \mathbb{R} \setminus \{0\} \Leftrightarrow X_1 = \begin{bmatrix} -\alpha \\ \alpha \end{bmatrix}$ se $\alpha = 1 \Rightarrow X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$(A - \lambda_2 I) X_2 = 0 \quad$ com $\lambda_2 = 2$

$$\begin{bmatrix} 0-\lambda_2 & -1 \\ 2 & 3-\lambda_2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 2 & 3-2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \left[\begin{array}{cc|c} -2 & -1 & 0 \\ 2 & 1 & 0 \end{array}\right]_{L_2 \leftarrow L_2 + L_1} \Leftrightarrow \left[\begin{array}{cc|c} -2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{cases} -2x - y = 0 \\ 0 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} -2x = y \\ 0 = 0 \end{cases} \Leftrightarrow \begin{cases} x = -\frac{1}{2} y \\ y \in \mathbb{R} \setminus \{0\} \end{cases} \Leftrightarrow \begin{cases} x = -\frac{1}{2} \alpha \\ y = \alpha \end{cases} \text{com } \alpha \in \mathbb{R} \setminus \{0\}$$

$$\Leftrightarrow X_2 = \begin{bmatrix} -\frac{1}{2}\alpha \\ \alpha \end{bmatrix} ; \text{ S } \alpha = 1 \Rightarrow X_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$