Exame de Recurso de <u>Matemática I</u>, 9 Janeiro de 2011

1) $\begin{cases} x + 2y + 3z = 9 \\ 3x - z = 3 \\ 2x - y + z = 8 \end{cases}$

a) $Ax = b \Leftrightarrow \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \\ 2 & -1 & 1 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 9 \\ 3 \\ 8 \end{bmatrix}}_{b}$

b) $|A| = \left|\begin{matrix} 1 & 2 & 3 \\ 3 & 0 & -1 \\ 2 & -1 & 1 \end{matrix}\right| \begin{matrix} 1 & 2 \\ 3 & 0 \\ 2 & -1 \end{matrix} = 0 - 4 - 9 - (0 + 1 + 6) = -13 - 7 = -2$

$\begin{matrix} - & - & - & + & + & + \end{matrix}$

A é não-singular porque $|A| = -20 \neq 0$.

c) calcular A^{-1} pelo método de Gauss-Jordan

$[A \,|\, I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 0 & -1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \underset{\substack{L_2 \leftarrow L_2 - 3L_1 \\ L_3 \leftarrow L_3 - 2L_1}}{\Leftrightarrow} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -6 & -10 & -3 & 1 & 0 \\ 0 & -5 & -5 & -2 & 0 & 1 \end{array}\right]$

$\underset{\Leftrightarrow}{\substack{L_2 \leftarrow -\frac{1}{6} L_2 \\ L_3 \leftarrow -\frac{1}{5} L_3}} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{3} & \frac{1}{2} & -\frac{1}{6} & 0 \\ 0 & 1 & 1 & \frac{2}{5} & 0 & -\frac{1}{5} \end{array}\right] \underset{\Leftrightarrow}{L_3 - L_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5/3 & 1/2 & -1/6 & 0 \\ 0 & 0 & -\frac{2}{3} & -1/10 & \frac{1}{6} & -1/5 \end{array}\right] \underset{\Leftrightarrow}{L_3 \leftarrow -\frac{3}{2} L_3}$

$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5/3 & 1/2 & -1/6 & 0 \\ 0 & 0 & 1 & 3/20 & -1/4 & 3/10 \end{array}\right] \underset{\Leftrightarrow}{\substack{L_1 \leftarrow L_1 - 3L_3 \\ L_2 \leftarrow L_2 - \frac{5}{3} L_3}} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 11/20 & 3/4 & -9/10 \\ 0 & 1 & 0 & 1/4 & 3/12 & -1/2 \\ 0 & 0 & 1 & 3/20 & -1/4 & 3/10 \end{array}\right]$

$$L_1 \leftarrow L_1 - 2L_2 \iff \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/20 & 1/4 & 1/10 \\ 0 & 1 & 0 & 1/4 & 1/4 & -1/2 \\ 0 & 0 & 1 & 3/20 & -1/4 & 3/10 \end{array}\right] = \left[I \mid A^{-1} \right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{20} & \frac{1}{4} & \frac{1}{10} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \\ \frac{3}{20} & -\frac{1}{4} & \frac{3}{10} \end{bmatrix}$$

(Em alternativa A^{-1} pode ser calculado pelo método da matriz adjunta)

d) $Ax = b \iff A^{-1}Ax = A^{-1}b \iff Ix = A^{-1}b \iff x = A^{-1}b$

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/20 & 1/4 & 1/10 \\ 1/4 & 1/4 & -1/2 \\ 3/20 & -1/4 & 3/10 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 3 \\ 8 \end{bmatrix} = \begin{bmatrix} (1/20)(9) + (1/4)(3) + (1/10)(8) \\ (1/4)(9) + (1/4)(3) + (-1/2)(8) \\ (3/20)(9) + (-1/4)(3) + (3/10)(8) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{20} + \frac{15}{20} + \frac{16}{20} \\ \frac{9}{4} + \frac{3}{4} - \frac{16}{4} \\ \frac{27}{20} - \frac{15}{20} + \frac{48}{20} \end{bmatrix} = \begin{bmatrix} \frac{40}{20} \\ -\frac{4}{4} \\ \frac{60}{20} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \iff \begin{cases} x = 2 \\ y = -1 \\ z = 3 \end{cases}$$

2) $A = \begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$

a) A é normal se $A^T A = A A^T$

$$A^T A = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 3 & -1 \\ 3 & -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 6 \\ -4 & 14 & -14 \\ 6 & -14 & 17 \end{bmatrix}$$

$$A A^T = \begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ -2 & 3 & -1 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 17 & -12 & 8 \\ -12 & 13 & -7 \\ 8 & -7 & 5 \end{bmatrix}$$

A não é uma matriz normal porque $A^T A \neq A A^T$.

b) $\text{tra}(A) = 2 + 3 + 2 = 7$

c) $|A - \lambda I| = 0 \Leftrightarrow \left| \begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$

$\Leftrightarrow \begin{vmatrix} 2-\lambda & -3 & 3 \\ 0 & 3-\lambda & -2 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0 \Leftrightarrow (2-\lambda)(3-\lambda)(2-\lambda) + (-3)(-2)(0) +$

$+ (3)(0)(-1) - (0)(3-\lambda)(3) - (-1)(-2)(2-\lambda) - (2-\lambda)(0)(-3) = 0$

$\Leftrightarrow (2-\lambda)(3-\lambda)(2-\lambda) - 2(2-\lambda) = 0$

$$\Leftrightarrow (2-\lambda)\left[(3-\lambda)(2-\lambda)-2\right]=0$$

$$\Leftrightarrow (2-\lambda)\left(6-3\lambda-2\lambda+\lambda^2-2\right)=0$$

$$\Leftrightarrow (2-\lambda)\left(\lambda^2-5\lambda+4\right)=0$$

$$\Leftrightarrow (2-\lambda)(\lambda-4)(\lambda-1)=0$$

$$\Leftrightarrow \lambda=2 \vee \lambda=4 \vee \lambda=1$$

$$\lambda(A)=\{1;2;4\}$$

C.A. factorizar $\lambda^2-5\lambda+4$

$$\lambda=\frac{5\pm\sqrt{25-4(1)(4)}}{2}$$

$$\Leftrightarrow \lambda=\frac{5\pm\sqrt{9}}{2} \Leftrightarrow \lambda=\frac{5\pm 3}{2}$$

$$\Leftrightarrow \lambda=\frac{8}{2}=4 \vee \lambda=\frac{2}{2}=1$$

$$\lambda^2-5\lambda+4=(\lambda-4)(\lambda-1)$$

d) $\lambda=4$

$$(A-\lambda I)X=0 \Leftrightarrow (A-4I)X=0$$

$$\Leftrightarrow \begin{bmatrix} 2-4 & -3 & 3 \\ 0 & 3-4 & -2 \\ 0 & -1 & 2-4 \end{bmatrix}\cdot\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -2 & -3 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}\cdot\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} -2 & -3 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \underset{\Leftrightarrow}{L_3 \leftarrow L_3 - L_2} \left[\begin{array}{ccc|c} -2 & -3 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{cases} -2x-3y+3z= \\ -y-2z=0 \\ 0=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} -2x - 3(-2z) + 3z = 0 \\ y = -2z \\ z \in \mathbb{R} \setminus \{0\} \end{cases} \Leftrightarrow \begin{cases} -2x + 6z + 3z = 0 \\ y = -2z \\ z \in \mathbb{R} \setminus \{0\} \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \frac{9}{2} z \\ y = -2z \\ z \in \mathbb{R} \setminus \{0\} \end{cases} \Leftrightarrow X = \begin{bmatrix} \frac{9}{2}\alpha \\ -2\alpha \\ \alpha \end{bmatrix} , \quad \alpha \in \mathbb{R} \setminus \{0\}$$

$$\text{si } \alpha = 2 \Rightarrow X = \begin{bmatrix} 9 \\ -4 \\ 2 \end{bmatrix}$$

3) a) $\int_0^2 -2x + x^2 + 2x \, dx$

$$\int -2x + x^2 + 2x \, dx = - \int 2x \, dx + \int x^2 dx + 5 \int x^3 dx$$

$$= x^2 + \frac{x^3}{3} + 5 \times \frac{x^4}{4} + c$$

$$\int_0^2 (-2x + x^2 + 5 \times x) dx = \left[x^2 + \frac{x^3}{3} + 5 \times \frac{x^4}{4} \right]_0^2 = \left(2^2 + \frac{2^3}{3} + 5 \times \frac{2^4}{4} \right) - (0)$$

$$= \left(4 + \frac{8}{3} + 5 \times \frac{16}{4} \right) - 0 = -4 + \frac{8}{3} + 20 = \frac{8}{3} + 16$$

$$= \frac{8 + 48}{3} = \frac{56}{3}$$

b) $$\int_1^2 \frac{x^2+x+1}{x}\,dx = \int_1^2 \frac{x^2}{x^2} + \frac{x}{x} + \frac{1}{x}\,dx$$

$$= \int_1^2 \frac{x^2}{x} + \frac{x}{x} + \frac{1}{x}\,dx$$

$$\int x + 1 + \frac{1}{x}\,dx = \frac{x^2}{2} + x + \ln|x| + c$$

$$\int_1^2 \frac{x^2+x+1}{x}\,dx = \left[\frac{x^2}{2} + x + \ln|x|\right]_1^2 = \left(\frac{2^2}{2} + 2 + \ln(2)\right) - \left(\frac{1^2}{2} + 1 + \ln(1)\right)$$

$$= 4 + \ln(2) - \frac{3}{2}$$

$$= \frac{5}{2} + \ln(2)$$

c) $$\int x e^{-3x}\,dx = -\frac{1}{3}\int -3e^{-3x}\,x\,dx$$

Vamos primitivar por partes $\int -3e^{-3x}\,x\,dx$

$$\boxed{\int f(x)g(x)\,dx = F(x)g(x) - \int F(x)g'(x)\,dx}$$

Consideramos que $f(x) = -3e^{-3x}$ e $g(x) = x$ pois $F(x) = e^{-3x}$

$$\int -3e^{-3x}\,dx = e^{-3x}x - \int e^{-3x} \times x'\,dx = e^{-3x}x - \int e^{-3x}\,dx$$

$$= e^{-3x}x - \left(-\frac{1}{3}\right)\int -3e^{-3x}\,dx = e^{-3x}x + \frac{1}{3}e^{-3x} + c_1$$

$$= e^{-3x}\left(x + \frac{1}{3}\right) + c_1$$

$$\int x e^{-3x}\, dx = -\frac{1}{3}\int -3e^{-3x}\, dx = -\frac{1}{3}\left[e^{-3x}\left(x+\frac{1}{3}\right)+c_1\right]$$

$$= -\frac{e^{-3x}}{3}\left(x+\frac{1}{3}\right) - \frac{c_1}{3}$$

$$= -\frac{e^{-3x}}{3}\left(x+\frac{1}{3}\right) + C$$

$$\int_{-1}^{0} x e^{-3x}\, dx = \left[-\frac{e^{-3x}}{3}\left(x+\frac{1}{3}\right)\right]_{-1}^{0} = -\frac{e^{0}}{3}\left(0+\frac{1}{3}\right) - \left[-\frac{e^{3}}{3}\left(-1+\frac{1}{3}\right)\right]$$

$$= \left(-\frac{1}{3}\right)\left(\frac{1}{3}\right) + \frac{e^3}{3}\left(-\frac{2}{3}\right) = -\frac{1}{9} - \frac{2e^3}{9}$$

$$= -\frac{1}{9}\left(1-2e^3\right)$$

d) $$\int_1^4 \frac{1}{\sqrt{x}\,(\sqrt{x}+1)^2}\, dx = 2\int_1^4 \frac{1}{(\sqrt{x}+1)^2}\cdot\frac{1}{2\sqrt{x}}\, dx$$

Substituição: $u = \sqrt{x}+1$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$\Leftrightarrow du = \frac{1}{2\sqrt{x}}\, dx$$

$u(1) = \sqrt{1}+1 = 2$

$u(4) = \sqrt{4}+1 = 3$

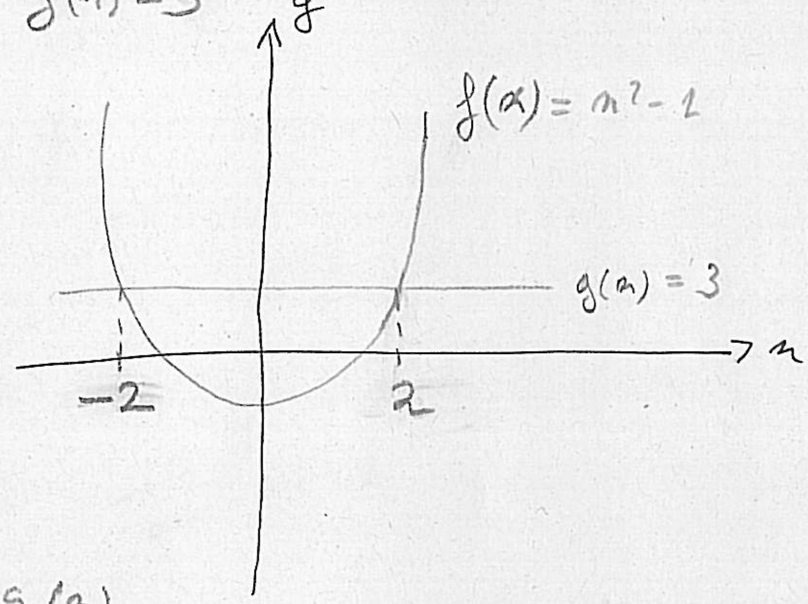
$$= 2\int_2^3 \frac{1}{u^2}\cdot du = 2\int_2^3 u^{-2}\, du$$

$$= 2\left[\frac{u^{-2+1}}{-2+1}\right]_2^3 = 2\left[\frac{u^{-1}}{-1}\right]_2^3 = 2\left[\frac{-1}{u}\right]_2^3 = 2\left[\left(-\frac{1}{3}\right)-\left(-\frac{1}{2}\right)\right]$$

$$= 2\left(-\frac{1}{3}+\frac{1}{2}\right) = 2\left(-\frac{2}{6}+\frac{3}{6}\right) = 2\times\frac{1}{6} = \frac{1}{3}$$

4) $f(x) = x^2 - 1$ e $g(x) = 3$

a)



$f(x) = g(x)$

$\Leftrightarrow x^2 - 1 = 3 \Leftrightarrow x^2 = 4 \Leftrightarrow x = \pm\sqrt{4} \Leftrightarrow x = -2 \vee x = 2$

b)

$$\text{Área} = \int_{-2}^{2} g(x) - f(x)\, dx = \int_{-2}^{2} 3 - (x^2 - 1)\, dx = \int_{-2}^{2} -x^2 + 4\, dx$$

$$= \left[-\frac{x^2}{2} + 4x\right]_{-2}^{2} = \left(-\frac{2^2}{2} + 4(2)\right) - \left(-\frac{(-2)^2}{2} + 4(-2)\right)$$

$$= -2 + 8 - (-2 - 8) = 6 - (-10) = 2 + 10 = 16$$

c)

$$V = \int_{-2}^{2} \pi (g(x) - f(x))^2\, dx = \int_{-2}^{2} \pi\, [3 - (x^2 - 2)]^2\, dx = \int_{-2}^{2} \pi [4 - x^2)^2\, dx$$

$$= \pi \int_{-2}^{2} 16 - 8x^2 + x^4\, dx = \left[16x - \frac{8x^2}{2} + \frac{x^5}{5}\right]_{-2}^{2} =$$

$$= \left(16 \times 2 - \frac{8 \times 2^2}{2} + \frac{2^5}{5}\right) - \left(16(-2) - \frac{8(-2)^2}{2} + \frac{(-2)^5}{5}\right) = \left(32 - 16 + \frac{32}{5}\right) - \left(-32 + 16 \ldots\right.$$

$$= 32 + \frac{64}{5} = \frac{160 + 64}{5} = \frac{224}{5}$$