Proposta de correcção do exame de Mat. I de 18/01/2011 - E.E.R.

1 - a)
$$\begin{cases} x + 2y + 3z = 9 \\ 2x - y + z = 8 \\ 3x - z = 3 \end{cases} \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right)$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{vmatrix} = +1 + 0 - 6 - (-9) - 0 - (-6) = 10 \neq 0.$$

$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & -1 & 1 \end{array}$$

logo o sistema tem solução única

1 - b)
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right) \xrightarrow[l_2 \leftarrow l_2 - 2l_1]{l_3 \leftarrow l_3 - 3l_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array}\right) \xrightarrow[l_3 \leftarrow l_3/-2]{l_2 \leftarrow l_2/-5}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & 5 & 12 \end{array}\right) \xrightarrow{l_3 \leftarrow l_3 - 3l_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 6 \end{array}\right)$$

$$\begin{cases} x + 2y + 3z = 9 \\ y + z = 2 \\ 2z = 6 \end{cases} \Leftrightarrow \begin{cases} x + 2y + 3z = 9 \\ y + 3 = 2 \\ z = 3 \end{cases} \Leftrightarrow \begin{cases} x - 2 + 9 = 9 \\ y = -1 \\ z = 3 \end{cases} \Leftrightarrow \begin{cases} x = 2 \\ y = -1 \\ z = 3 \end{cases}$$

2 → $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

2- a) A é simétrica, pois $A^T = A$.

b) $Tr(A) = 1 + 1 = 2$ (soma dos elementos da diagonal principal)

c) Polinómio característico: $P(\lambda)$

$$P(\lambda) = \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 9 = 1 - 2\lambda + \lambda^2 - 9$$

$$= \lambda^2 - 2\lambda - 8$$

$$P(\lambda) = 0 \Leftrightarrow \lambda^2 - 2\lambda - 8 = 0 \Leftrightarrow \lambda = \frac{2 \pm \sqrt{4 + 4 \times 8}}{2}$$

$$\Leftrightarrow \lambda = \frac{2 \pm 6}{2} \Leftrightarrow \lambda = 4 \vee \lambda = -2$$

d) $\underline{\lambda = 4}$:

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 4 \begin{bmatrix} x \\ y \end{bmatrix} \Leftrightarrow \begin{bmatrix} x + 3y \\ 3x + y \end{bmatrix} = \begin{bmatrix} 4x \\ 4y \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x + 3y = 4x \\ 3x + y = 4y \end{cases} \Leftrightarrow \begin{cases} -3x + 3y = 0 \\ 3x - 3y = 0 \end{cases} \Leftrightarrow \begin{cases} x = y \\ x = y \end{cases}$$

sol: $\{(x, y) \in \mathbb{R}^2 : x = y\}$ Vector próprio: $(1, 1)$

3) $y = \dfrac{\sqrt{x^2+7}}{x-2}$

a) $y' = \dfrac{\left(\sqrt{x^2+7}\right)'(x-2) - \sqrt{x^2+7}\,(x-2)'}{(x-2)^2} = \dfrac{x\,(x^2+7)^{-\frac{1}{2}}(x-2) - \sqrt{x^2+7}}{(x-2)^2}$

$= \dfrac{x\,(x-2)}{\sqrt{x^2+7}\,(x-2)^2} - \dfrac{\sqrt{x^2+7}}{(x-2)^2} = \dfrac{x}{\sqrt{x^2+7}\,(x-2)} - \dfrac{\sqrt{x^2+7}}{(x-2)^2}$

$\boxed{\text{C.A.}}$ $\left(\sqrt{x^2+7}\right)' = \left[(x^2+7)^{\frac{1}{2}}\right]' = \frac{1}{2}(x^2+7)^{\frac{1}{2}-1}\cdot(x^2+7)'$

$= \frac{1}{\cancel{2}}(x^2+7)^{-\frac{1}{2}}(\cancel{2}x) = x\,(x^2+7)^{-\frac{1}{2}} = \dfrac{x}{\sqrt{x^2+7}}$

b) $y = mx + b$, onde $m = y'(3)$ pois a recta passa em $P(3;$

$y'(3) = \dfrac{3}{\sqrt{3^2+7}\,(3-2)} - \dfrac{\sqrt{3^2+7}}{(3-2)^2} = \dfrac{3}{\sqrt{16}\,(1)} - \dfrac{\sqrt{16}}{(1)^2} = \dfrac{3}{4} - 4$

$= \dfrac{3}{4} - \dfrac{16}{4} = -\dfrac{13}{4} = m \Rightarrow y = -\dfrac{13}{4}x + b$

Se $x = 3 \Rightarrow y = 4$: $4 = -\frac{13}{4}\times 3 + b \Leftrightarrow 4 = -\frac{39}{4} + b \Leftrightarrow b = 4 + \frac{39}{4} = \frac{55}{4}$

$\boxed{y = -\frac{13}{4}x + \frac{55}{4}}$

c) $y = m_1 x + b$ com $m_1 = \dfrac{1}{m} = \dfrac{1}{-\frac{13}{4}} = -\dfrac{4}{13}$

$y = -\frac{4}{13}x + b$ como passa também no ponto $P(3;4)$

$4 = -\frac{4}{13}(3) + b \Leftrightarrow 4 = -\frac{12}{13} + b \Leftrightarrow b = 4 + \frac{12}{13}$

$\Leftrightarrow b = \dfrac{52+12}{13} = \dfrac{64}{13}$

$\boxed{y = -\frac{4}{13}x + \frac{64}{13}}$

3

(3)

4-a) $$\int_{-1}^{1} 2x + x^2 - 5x^4 \, dx = \left[x^2 + \frac{x^3}{3} - \frac{\cancel{5}x^5}{\cancel{5}} \right]_{-1}^{1} =$$

$$= \cancel{1} + \frac{1}{3} - \cancel{1} - \left(1 - \frac{1}{3} - (-1)\right)$$

$$= \frac{1}{3} - 1 + \frac{1}{3} - 1 = \frac{2}{3} - 2$$

$$= \frac{2}{3} - \frac{6}{3} = -\frac{4}{3}$$

4-b) $$\int_{1}^{2} \frac{x+1}{x^2} \, dx = \int_{1}^{2} \frac{x}{x^2} + \frac{1}{x^2} \, dx =$$

$$= \int_{1}^{2} \frac{1}{x} + x^{-2} \, dx = \left[\ln|x| - \frac{1}{x} \right]_{1}^{2} =$$

$$= \ln 2 - \frac{1}{2} - \left(\overbrace{\ln 1}^{=0} - \frac{1}{1} \right)$$

$$= \ln 2 - \frac{1}{2} + 1 = \ln 2 + \frac{1}{2}$$

C.A.

$$\int x^{-2} dx = \frac{x^{-1}}{-1} + c = -\frac{1}{x} + c$$

4-c) $$\cancel{\int_{-2}^{0} \frac{x^2}{(x^3-2)^2} \, dx = \int_{-2}^{0} x^2 (\cancel{3}x^3 - 2)^{-2} \, dx =}$$

$$\cancel{= \frac{1}{3} \int_{-2}^{0} 3x^2 (x^3-2)^{-2} \, dx = \frac{1}{3} \left[\frac{(x^3-2)^{-1}}{-1} \right]_{-2}^{0} =}$$

4-c) (cont.)

$$= \frac{1}{3}\left[-\frac{1}{x^3-2}\right]_{-2}^{0} = \frac{1}{3}\left[-\frac{1}{0^3-2} + \frac{1}{(-2)^3-2}\right]$$

$$= \frac{1}{3}\left(+\frac{1}{+2} + \frac{1}{-8-2}\right) = \frac{1}{3}\left(\frac{1}{2} - \frac{1}{10}\right)$$

$$= \frac{1}{3}\left(\frac{5}{10} - \frac{1}{10}\right) = \frac{1}{3}\left(\frac{4}{10}\right) = \frac{4}{30} = \frac{2}{15}$$

4-d) $\int_1^2 x \ln(x^2)\, dx = \left[\ln(x^2)\cdot\frac{x^2}{2}\right]_1^2 - \int_1^2 \frac{2}{x}\cdot\frac{x^2}{2}\, dx$

$$= \left(\ln(2^2) \times \frac{2^2}{2} - \underbrace{\ln(1^2)}_{=0}\cdot\frac{1^2}{2}\right) - \int_1^2 x\, dx$$

$$= 2\ln(2) \times \frac{4}{2} - 0 - \left[\frac{x^2}{2}\right]_1^2 =$$

$$= 4\ln 2 - \left(\frac{2^2}{2} - \frac{1^2}{2}\right) = 4\ln 2 - \frac{3}{2}$$

$\boxed{\text{Nota:}}$ $\int f g' = fg - \int f' g$, onde $f = \ln(x^2)$;

$g' = x$; $g = \frac{x^2}{2}$

$\to f' = \frac{(x^2)'}{x^2} = \frac{2x}{x^2} = \frac{2}{x}$

4 - d) $\int_1^2 x \ln(x^2)\,dx$

<u>Nota:</u> vamos primitivar por partes (cap.7, p. 10)

$$\int f(x)\cdot g(x)\,dx = F(x)\cdot g(x) - \int F(x)\cdot g'(x)\,dx$$

com $F(x) = \int f(x)\,dx$ (primitiva de $f(x)$)

e $g'(x)$ a derivada de $g(x)$

Se fizermos $f(x) = x$ e $g(x) = \ln(x^2)$

Vamos ter $F(x) = \int x\,dx = \frac{x^2}{2}$

e $g'(x) = (\ln(x^2))' = \frac{1}{x^2}\cdot(x^2)' = \frac{2x}{x^2} = \frac{2}{x}$

logo

$$\int x\ln(x^2)\,dx = \frac{x^2}{2}\ln(x^2) - \int \frac{x^2}{2}\cdot\frac{2}{x}\,dx =$$

$$= \frac{x^2}{2}\ln(x^2) - \int x\,dx = \frac{x^2}{2}\ln(x^2) - \frac{x^2}{2} + C$$

assim

$$\int_1^2 x\ln(x^2)\,dx = \left[\frac{x^2}{2}\ln(x^2) - \frac{x^2}{2}\right]_1^2 =$$

$$= \left(\frac{2^2}{2}\ln(2^2) - \frac{2^2}{2}\right) - \left(\frac{1^2}{2}\ln(1^2) - \frac{1^2}{2}\right)$$

$$= 2\ln(4) - 2 - \frac{1}{2}\ln(1) + \frac{1}{2} = 2\ln(4) - 2 - \frac{1}{2}(0) + \frac{1}{2}$$

$$= 2\ln(4) - 2 + \frac{1}{2} = 2\ln(4) - \frac{3}{2}$$

4 - c) $\int_{-2}^{0} \frac{x^2}{(x^3-2)^2}\,dx$

C.A.

Mudança de variável (cap. 9): $u = (x^3-2)$

$\frac{du}{dx} = 3x^2 \Leftrightarrow du = 3x^2\,dx$

$u(0) = 0^3 - 2 = -2$

$u(-2) = (-2)^3 - 2 = -8 - 2 = -10$

$$\int_{-2}^{0} \frac{x^2}{(x^3-2)^2}\,dx = \frac{1}{3}\int_{-2}^{0} \frac{1}{(x^3-2)^2} \cdot 3x^2\,dx = \frac{1}{3}\int_{-10}^{-2} \frac{1}{u^2}\,du$$

$$= \frac{1}{3}\int_{-10}^{-2} u^{-2}\,du = \frac{1}{3}\left[\frac{u^{-2+1}}{-2+1}\right]_{-10}^{-2} = \frac{1}{3}\left[-\frac{1}{u}\right]_{-10}^{-2}$$

$$= \frac{1}{3}\left[\left(-\frac{1}{-2}\right) - \left(-\frac{1}{-10}\right)\right] = \frac{1}{3}\left(\frac{1}{2} - \frac{1}{10}\right) = \frac{1}{3}\left(\frac{5}{10} - \frac{1}{10}\right)$$

$$= \frac{1}{3}\left(\frac{4}{10}\right) = \frac{4}{30} = \frac{2}{15}$$