O Método da Transformada de Laplace

4.1. Cálculo da Transformada de Laplace

Exercício 4.1.1

a) 1

$$\begin{aligned}\mathcal{L}\{1\} = \int_0^{+\infty} e^{-st}dt &= \lim_{r\to+\infty} -\frac{e^{-st}}{s}\Big|_0^r \\ &= \lim_{r\to+\infty}\left(-\frac{e^{-sr}}{s}+\frac{1}{s}\right) \\ &= \frac{1}{s}, \quad s>0.\end{aligned}$$

b) e^{at}

$$\begin{aligned}\mathcal{L}\left\{e^{at}\right\} = \int_0^{+\infty} e^{-st}e^{at}dt &= \int_0^{+\infty} e^{(a-s)t}dt \\ &= \lim_{r\to+\infty} \frac{e^{(a-s)t}}{a-s}\Big|_0^r \\ &= \lim_{r\to+\infty}\left(\frac{e^{(a-s)r}}{a-s}-\frac{1}{a-s}\right) \\ &= \frac{1}{s-a}, \quad s>a.\end{aligned}$$

c) $at+b$

$$\begin{aligned}\mathcal{L}\{at+b\} = \int_0^{+\infty} e^{-st}(at+b)\,dt &= \lim_{r\to+\infty}\int_0^r e^{-st}(at+b)\,dt \\ &= \lim_{r\to+\infty}\left(-\frac{at+b}{s}e^{-st}\Big|_0^r + \frac{1}{s}\int_0^r ae^{-st}\right) \\ &= \lim_{r\to+\infty}\left(-\frac{ar+b}{s}e^{-sr}+\frac{b}{s}-\left(\frac{a}{s^2}e^{-st}\right)\Big|_0^r\right) \\ &= \lim_{r\to+\infty}\left(-\frac{ar+b}{s}e^{-sr}+\frac{b}{s}-\left(\frac{ae^{-sr}}{s^2}-\frac{a}{s^2}\right)\right) \\ &= \frac{b}{s}+\frac{a}{s^2} \quad s>0.\end{aligned}$$

d) e^{at+b}

$$\begin{aligned}\mathcal{L}\left\{e^{at+b}\right\} = \int_0^{+\infty} e^{-st}e^{at+b}dt &= e^b\int_0^{+\infty} e^{(a-s)t}dt \\ &= \frac{e^b}{s-a} \quad s>a \text{ (por b))}\end{aligned}$$

e) $\cos^2(at)$

$$\begin{aligned}
\mathcal{L}\left\{\cos^2(at)\right\} &= \int_0^{+\infty} e^{-st}\cos^2(at)\,dt \\
&= \int_0^{+\infty} \frac{1+\cos(2at)}{2}e^{-st}dt \\
&= \lim_{r\to+\infty}\left(\frac{1}{2}\int_0^r e^{-st}dt + \frac{1}{2}\int_0^r e^{-st}\cos(2at)\,dt\right) \\
&= \lim_{r\to+\infty}\left(-\left.\frac{e^{-st}}{2s}\right|_0^r + \frac{1}{2}\int_0^r e^{-st}\cos(2at)\,dt\right)
\end{aligned}$$

mas, usando primitivação por partes,

$$\begin{aligned}
\int e^{-st}\cos(2at)\,dt &= -\frac{e^{-st}\cos(2at)}{s} - \frac{2a}{s}\int e^{-st}\sin(2at)\,dt \\
&= -\frac{e^{-st}\cos(2at)}{s} - \frac{2a}{s}\left(-\frac{e^{-st}\sin(2at)}{s} + \frac{2a}{s}\int e^{-st}\cos(2at)\,dt\right) \\
&= -\frac{e^{-st}\cos(2at)}{s} + \frac{2ae^{-st}\sin(2at)}{s^2} - \frac{4a^2}{s^2}\int e^{-st}\cos(2at)\,dt
\end{aligned}$$

ou seja,

$$\int e^{-st}\cos(2at)\,dt = \frac{2a\sin(2at) - s\cos(2at)}{s^2+4a^2}e^{-st}$$

e portanto, retomando o cálculo da transformada de Laplace da função $\cos^2(at)$ obtemos

$$\begin{aligned}
\mathcal{L}\left\{\cos^2(at)\right\} &= \lim_{r\to+\infty}\left(-\left.\frac{e^{-st}}{2s}\right|_0^r + \frac{1}{2}\int_0^r e^{-st}\cos(2at)\,dt\right) \\
&= \lim_{r\to+\infty}\left(-\left.\frac{e^{-st}}{2s}\right|_0^r + \left.\frac{2a\sin(2at) - s\cos(2at)}{2(s^2+4a^2)}e^{-st}\right|_0^r\right) \\
&= \lim_{r\to+\infty}\left(-\frac{e^{-sr}}{2s} + \frac{1}{2s} + \frac{2a\sin(2ar) - s\cos(2ar)}{2(s^2+4a^2)}e^{-sr} + \frac{s}{2(s^2+4a^2)}\right) \\
&= \frac{1}{2s} + \frac{s}{2(s^2+4a^2)} \qquad s>0.
\end{aligned}$$

f) $\cos(at)\sin(bt)$

$$\begin{aligned}
\mathcal{L}\left\{\cos(at)\sin(bt)\right\} &= \int_0^{+\infty} e^{-st}\cos(at)\sin(bt)\,dt \\
&= \int_0^{+\infty} \frac{\sin(bt+at)+\sin(bt-at)}{2}e^{-st}dt \\
&= \lim_{r\to+\infty}\left(\int_0^r \frac{\sin(bt+at)+\sin(bt-at)}{2}e^{-st}dt\right) \\
&= \ldots \\
&= \lim_{r\to+\infty}\left(\frac{-a\cos(at+bt) - b\cos(at+bt) - s\sin(at+bt)}{2\left(s^2+(a+b)^2\right)}e^{-st}\right. \\
&\qquad \left.\left. + \frac{a\cos(at-bt) - b\cos(at-bt) + s\sin(at-bt)}{2\left(s^2+(a-b)^2\right)}e^{-st}\right)\right|_0^r \\
&= \frac{a+b}{2\left(s^2+(a+b)^2\right)} - \frac{a-b}{2\left(s^2+(a-b)^2\right)} \qquad s>0.
\end{aligned}$$

g) $\cos(at)$

$$\begin{aligned}\mathcal{L}\{\cos(at)\} &= \int_0^{+\infty} e^{-st}\cos(at)\,dt \\ &= \lim_{r\to+\infty}\int_0^r \cos(at)\,e^{-st}dt \\ &= \lim_{r\to+\infty}\left.\frac{a\sin(at)-s\cos(at)}{s^2+a^2}e^{-st}\right|_0^r \\ &= \frac{s}{s^2+a^2} \qquad s>0.\end{aligned}$$

h) $\sin^2(at)$

$$\begin{aligned}\mathcal{L}\left\{\sin^2(at)\right\} &= \int_0^{+\infty} e^{-st}\sin^2(at)\,dt \\ &= \int_0^{+\infty}\frac{1-\cos(2at)}{2}e^{-st}dt \\ &= \lim_{r\to+\infty}\left(\frac{1}{2}\int_0^r e^{-st}dt-\frac{1}{2}\int_0^r e^{-st}\cos(2at)\,dt\right) \\ &= \lim_{r\to+\infty}\left.\left(-\frac{e^{-st}}{2s}-\frac{1}{2}\frac{2a\sin(2at)-s\cos(2at)}{s^2+4a^2}e^{-st}\right)\right|_0^r \\ &= \frac{1}{2s}-\frac{1}{2}\frac{s}{s^2+4a^2} \qquad s>0.\end{aligned}$$

i) $\sin(at)$

$$\begin{aligned}\mathcal{L}\{\sin(at)\} &= \int_0^{+\infty} e^{-st}\sin(at)\,dt \\ &= \lim_{r\to+\infty}\int_0^r \sin(at)\,e^{-st}dt \\ &= \lim_{r\to+\infty}\left.\frac{-a\cos(at)-s\sin(at)}{s^2+a^2}e^{-st}\right|_0^r \\ &= \frac{a}{s^2+a^2} \qquad s>0.\end{aligned}$$

j) $\sinh(at)=\frac{e^{at}-e^{-at}}{2}$

$$\begin{aligned}\mathcal{L}\{\sinh(at)\} &= \mathcal{L}\left\{\frac{e^{at}-e^{-at}}{2}\right\} \\ &= \int_0^{+\infty}\frac{e^{at}-e^{-at}}{2}e^{-st}dt \\ &= \frac{1}{2}\left(\int_0^{+\infty} e^{(a-s)t}dt-\int_0^{+\infty} e^{-(a+s)t}dt\right) \\ &= \lim_{r\to+\infty}\frac{1}{2}\left.\left(\frac{e^{(a-s)t}}{a-s}+\frac{e^{-(a+s)t}}{a+s}\right)\right|_0^r \\ &= \frac{1}{2}\left(\frac{1}{s-a}-\frac{1}{s+a}\right) \qquad s>|a| \\ &= \frac{a}{s^2-a^2}\end{aligned}$$

k) $\cosh(at) = \frac{e^{at}+e^{-at}}{2}$

$$\begin{aligned}
\mathcal{L}\{\cosh(at)\} &= \mathcal{L}\left\{\frac{e^{at}+e^{-at}}{2}\right\} \\
&= \int_0^{+\infty} \frac{e^{at}+e^{-at}}{2} e^{-st}dt \\
&= \frac{1}{2}\left(\int_0^{+\infty} e^{(a-s)t}dt + \int_0^{+\infty} e^{-(a+s)t}dt\right) \\
&= \lim_{r\to+\infty} \frac{1}{2}\left(\frac{e^{(a-s)t}}{a-s} - \frac{e^{-(a+s)t}}{a+s}\right)\Bigg|_0^r \\
&= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) \qquad s > |a| \\
&= \frac{s}{s^2-a^2}
\end{aligned}$$

l) (está repetido, é igual ao da alínea l))

m) $\left\{\begin{array}{ccc} 0 & , & 0<t<1 \\ 1 & , & t\geq 1 \end{array}\right.$

$$\mathcal{L}\left\{\left\{\begin{array}{ccc} 0 & , & 0<t<1 \\ 1 & , & t\geq 1 \end{array}\right.\right\} = \int_0^1 e^{-st}dt = -\frac{e^{-st}}{s}\Bigg|_0^1 = -\frac{e^{-s}}{s} + \frac{1}{s}$$

n) $\left\{\begin{array}{ccc} \sin t & , & 0<t<\pi \\ 0 & , & t\geq \pi \end{array}\right.$

$$\begin{aligned}
\mathcal{L}\left\{\left\{\begin{array}{ccc} \sin t & , & 0<t<\pi \\ 0 & , & t\geq \pi \end{array}\right.\right\} = \int_0^{\pi} \sin t e^{-st}dt &= \frac{-\cos t - s\sin t}{s^2+1}e^{-st}\Bigg|_0^{\pi} \\
&= \frac{1}{s^2+1}\left(e^{-\pi s}+1\right)
\end{aligned}$$

Exercício 4.1.2.

a) $y'' - 3y' + 2y = e^{3t}, \quad y(0) = 1, y'(0) = 1.$

$$\begin{aligned}
& y'' - 3y' + 2y = e^{3t} \\
\Leftrightarrow\ & \mathcal{L}\{y'' - 3y' + 2y\} = \mathcal{L}\{e^{3t}\} \\
\Leftrightarrow\ & \mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \frac{1}{s-3} \\
\Leftrightarrow\ & s^2Y - s - 1 - 3(sY-1) + 2Y = \frac{1}{s-3} \\
\Leftrightarrow\ & \left(s^2-3s+2\right)Y = \frac{1}{s-3} + s - 2 \\
\Leftrightarrow\ & (s-1)(s-2)Y = \frac{1}{s-3} + s - 2 \\
\Leftrightarrow\ & Y = \frac{1}{(s-1)(s-2)(s-3)} + \frac{1}{s-1}
\end{aligned}$$

$$\Leftrightarrow \quad \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s-2)(s-3)} + \frac{1}{s-1}\right\}$$
$$\Leftrightarrow \quad y = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s-2)(s-3)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$
$$\Leftrightarrow \quad y = \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}}{s-1} - \frac{1}{s-2} + \frac{\frac{1}{2}}{s-3}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$
$$\Leftrightarrow \quad y = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$
$$\Leftrightarrow \quad y = \frac{3e^t}{2} - e^{2t} + \frac{e^{3t}}{2}$$

b) $y'' - 5y' + 4y = e^{2t}, \quad y(0) = 1, y'(0) = -1.$

$$y'' - 5y' + 4y = e^{2t}$$
$$\Leftrightarrow \quad \mathcal{L}\{y'' - 5y' + 4y\} = \mathcal{L}\{e^{2t}\}$$
$$\Leftrightarrow \quad \mathcal{L}\{y''\} - 5\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \frac{1}{s-2}$$
$$\Leftrightarrow \quad s^2Y - s + 1 - 5(sY - 1) + 4Y = \frac{1}{s-2}$$
$$\Leftrightarrow \quad \left(s^2 - 5s + 4\right)Y = \frac{1}{s-2} + s - 6$$
$$\Leftrightarrow \quad (s-1)(s-4)Y = \frac{1}{s-2} + s - 6$$
$$\Leftrightarrow \quad Y = \frac{s^2 - 8s + 13}{(s-1)(s-2)(s-4)}$$
$$\Leftrightarrow \quad \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{s^2 - 8s + 13}{(s-1)(s-2)(s-4)}\right\}$$
$$\Leftrightarrow \quad y = \mathcal{L}^{-1}\left\{\frac{2}{s-1} - \frac{\frac{1}{2}}{s-2} - \frac{\frac{1}{2}}{s-4}\right\}$$
$$\Leftrightarrow \quad y = 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}$$
$$\Leftrightarrow \quad y = 2e^t - \frac{e^{2t}}{2} - \frac{e^{4t}}{2}$$

c) $y'' + 3y' + 7y = \cos t, \quad y(0) = 0, y'(0) = 2$

$$y'' + 3y' + 7y = \cos t$$
$$\Leftrightarrow \quad \mathcal{L}\{y'' + 3y' + 7y\} = \mathcal{L}\{\cos t\}$$
$$\Leftrightarrow \quad \mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 7\mathcal{L}\{y\} = \frac{s}{s^2+1}$$
$$\Leftrightarrow \quad s^2Y - 2 + 3sY + 7Y = \frac{s}{s^2+1}$$
$$\Leftrightarrow \quad \left(s^2 + 3s + 7\right)Y = \frac{s}{s^2+1} + 2$$
$$\Leftrightarrow \quad Y = \frac{2s^2 + s + 2}{(s^2+1)(s^2+3s+7)}$$

$$\Leftrightarrow \quad \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{2s^2+s+2}{(s^2+1)(s^2+3s+7)}\right\}$$

$$\Leftrightarrow \quad y = \mathcal{L}^{-1}\left\{\frac{\frac{2}{15}s+\frac{1}{15}}{s^2+1} + \frac{-\frac{2}{15}s+\frac{23}{15}}{s^2+3s+7}\right\}$$

$$\Leftrightarrow \quad y = \frac{2}{15}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{15}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{1}{15}\mathcal{L}^{-1}\left\{\frac{-2s+23}{s^2+3s+7}\right\}$$

$$\Leftrightarrow \quad y = \frac{2\cos t + \sin t}{15} + \frac{1}{15}\mathcal{L}^{-1}\left\{\frac{-2s+23}{s^2+3s+7}\right\}$$

mas

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{-2s+23}{s^2+3s+7}\right\} &= \mathcal{L}^{-1}\left\{\frac{-2\left(s+\frac{3}{2}\right)+26}{\left(s+\frac{3}{2}\right)^2+\frac{19}{4}}\right\} \\
&= -2\mathcal{L}^{-1}\left\{\frac{s+\frac{3}{2}}{\left(s+\frac{3}{2}\right)^2+\frac{19}{4}}\right\} + \frac{52}{\sqrt{19}}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{19}}{2}}{\left(s+\frac{3}{2}\right)^2+\frac{19}{4}}\right\} \\
&= -2e^{-\frac{3}{2}t}\cos\left(\frac{\sqrt{19}}{2}t\right) + \frac{52}{\sqrt{19}}e^{-\frac{3}{2}t}\sin\left(\frac{\sqrt{19}}{2}t\right)
\end{aligned}$$

Conclusão

$$\begin{aligned}
y &= \frac{2\cos t + \sin t}{15} + \frac{1}{15}\mathcal{L}^{-1}\left\{\frac{-2s+23}{s^2+3s+7}\right\} \\
&= \frac{2\cos t + \sin t}{15} + \frac{e^{-\frac{3}{2}t}}{15}\left(-2\cos\left(\frac{\sqrt{19}}{2}t\right) + \frac{52}{\sqrt{19}}\sin\left(\frac{\sqrt{19}}{2}t\right)\right)
\end{aligned}$$

d) $2y'' + y' - y = e^{3t}, \quad y(0) = 2, y'(0) = 0$

$$2y'' + y' - y = e^{3t}$$

$$\Leftrightarrow \quad \mathcal{L}\{2y'' + y' - y\} = \mathcal{L}\{e^{3t}\}$$

$$\Leftrightarrow \quad 2\mathcal{L}\{y''\} + \mathcal{L}\{y'\} - \mathcal{L}\{y\} = \mathcal{L}\{e^{3t}\}$$

$$\Leftrightarrow \quad 2\left(s^2Y - 2s\right) + sY - 2 - Y = \frac{1}{s-3}$$

$$\Leftrightarrow \quad \left(2s^2 + s - 1\right)Y = \frac{1}{s-3} + 4s + 2$$

$$\Leftrightarrow \quad Y = \frac{1}{(s-3)(2s^2+s-1)} + \frac{4s+2}{2s^2+s-1}$$

$$\Leftrightarrow \quad Y = \frac{4s^2-10s-5}{(s-3)(2s^2+s-1)}$$

$$\Leftrightarrow \quad \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{4s^2-10s-5}{(s-3)(2s^2+s-1)}\right\}$$

$$\Leftrightarrow \quad y = \mathcal{L}^{-1}\left\{\frac{\frac{3}{4}}{s+1} + \frac{\frac{1}{20}}{s-3} + \frac{\frac{6}{5}}{s-\frac{1}{2}}\right\}$$

$$\Leftrightarrow \quad y = \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{20}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{6}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-\frac{1}{2}}\right\}$$

$$\Leftrightarrow \quad y = \frac{3e^{-t}}{4} + \frac{e^{3t}}{20} + \frac{6e^{\frac{t}{2}}}{5}$$

e) $y'' - y = t, \quad y(0) = 1, y'(0) = 1$

$$\begin{aligned}
& y'' - y = t \\
\Leftrightarrow \quad & \mathcal{L}\left\{y'' - y\right\} = \mathcal{L}\{t\} \\
\Leftrightarrow \quad & \mathcal{L}\left\{y''\right\} - \mathcal{L}\{y\} = \frac{1}{s^2} \\
\Leftrightarrow \quad & s^2 Y - s - 1 - Y = \frac{1}{s^2} \\
\Leftrightarrow \quad & \left(s^2 - 1\right) Y = \frac{1}{s^2} + s + 1 \\
\Leftrightarrow \quad & (s-1)(s+1) Y = \frac{1}{s^2} + s + 1 \\
\Leftrightarrow \quad & Y = \frac{s^3 + s^2 + 1}{s^2 (s-1)(s+1)} \\
\Leftrightarrow \quad & \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{s^3 + s^2 + 1}{s^2 (s-1)(s+1)}\right\} \\
\Leftrightarrow \quad & y = \mathcal{L}^{-1}\left\{-\frac{1}{s^2} + \frac{\frac{3}{2}}{s-1} - \frac{\frac{1}{2}}{s+1}\right\} \\
\Leftrightarrow \quad & y = -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \\
\Leftrightarrow \quad & y = -t + \frac{3e^t - e^{-t}}{2}
\end{aligned}$$

f) $y'' + y = t^2 \sin t, \quad y(0) = y'(0) = 0$

$$\begin{aligned}
& y'' + y = t^2 \sin t \\
\Leftrightarrow \quad & \mathcal{L}\left\{y'' + y\right\} = \mathcal{L}\left\{t^2 \sin t\right\} \\
\Leftrightarrow \quad & \mathcal{L}\left\{y''\right\} + \mathcal{L}\{y\} = -\frac{d}{ds}\mathcal{L}\{t \sin t\} \\
\Leftrightarrow \quad & s^2 Y + Y = \frac{d^2}{ds^2}\mathcal{L}\{\sin t\} \\
\Leftrightarrow \quad & \left(s^2 + 1\right) Y = \frac{d^2}{ds^2}\left(\frac{1}{s^2 + 1}\right) \\
\Leftrightarrow \quad & Y = \frac{-2}{\left(s^2 + 1\right)^3} + \frac{8s^2}{\left(s^2 + 1\right)^4} \\
\Leftrightarrow \quad & \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{-2}{\left(s^2 + 1\right)^3} + \frac{8s^2}{\left(s^2 + 1\right)^4}\right\} \\
\Leftrightarrow \quad & y = -2\mathcal{L}^{-1}\left\{\frac{1}{\left(s^2 + 1\right)^3}\right\} + 8\mathcal{L}^{-1}\left\{\frac{s^2}{\left(s^2 + 1\right)^4}\right\}
\end{aligned}$$

veremos mais à frente como terminar a resolução deste exercício usando convolução.

Exercício 4.1.3. Ora

$$\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \int_0^{+\infty} \frac{e^{-st}}{\sqrt{t}} dt = \lim_{r \to +\infty} \int_0^r \frac{e^{-st}}{\sqrt{t}} dt.$$

Fazendo a mudança de variável

$$x = \sqrt{st} \Rightarrow x^2 = st, \quad dx = \frac{s dt}{2\sqrt{st}} \text{ e } 0 < x < \sqrt{sr}$$

e portanto

$$\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \lim_{r\to+\infty}\int_0^r \frac{e^{-st}}{\sqrt{t}}dt = \lim_{r\to+\infty}\frac{2}{\sqrt{s}}\int_0^{\sqrt{sr}} e^{-u^2}dx = \frac{2}{\sqrt{s}}\underbrace{\int_0^{+\infty} e^{-x^2}dx}_{=\frac{\sqrt{\pi}}{2}} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

4.2. Propriedades da Transformada de Laplace

Exercício 4.2.1.

a) te^t

$$\mathcal{L}\left\{te^t\right\} = -\frac{d}{ds}\left(\mathcal{L}\left\{e^t\right\}\right) = -\frac{d}{ds}\left(\frac{1}{s-1}\right) = \frac{1}{(s-1)^2} \quad s > 1$$

b) t^{13}

$$\mathcal{L}\left\{t^{13}\right\} = \frac{13!}{s^{14}} \quad s > 0$$

c) $t\sin(at)$

$$\mathcal{L}\left\{t\sin(at)\right\} = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \frac{2as}{(s^2+a^2)^2} \quad s > 0$$

d) $t^2\cos(at)$

$$\begin{aligned}\mathcal{L}\left\{t^2\cos(at)\right\} &= -\frac{d}{ds}\left(-\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)\right) \\ &= -\frac{d}{ds}\left(\frac{s^2-a^2}{(s^2+a^2)^2}\right) \\ &= \frac{2s^3-6a^2s}{(s^2+a^2)^3} \quad s > 0\end{aligned}$$

e) $te^{-t}+3t^2e^{-t}$

$$\begin{aligned}\mathcal{L}\left\{te^{-t}+3t^2e^{-t}\right\} &= \mathcal{L}\left\{te^{-t}\right\} + 3\mathcal{L}\left\{t^2e^{-t}\right\} \\ &= -\frac{d}{ds}\left(\frac{1}{s+1}\right) + 3\left(\frac{d^2}{ds^2}\left(\frac{1}{s+1}\right)\right) \\ &= \frac{1}{(s+1)^2} + \frac{6}{(s+1)^3} \\ &= \frac{s+7}{(s+1)^3} \quad s > -1\end{aligned}$$

f) $t^3e^{-3t}+4e^{-t}\cos(3t)$

$$\begin{aligned}\mathcal{L}\left\{t^3e^{-3t}+4e^{-t}\cos(3t)\right\} &= \mathcal{L}\left\{t^3e^{-3t}\right\} + 4\mathcal{L}\left\{e^{-t}\cos(3t)\right\} \\ &= -\frac{d^3}{ds^3}\left(\frac{1}{s+3}\right) + 4\frac{s+1}{(s+1)^2+3^2} \quad s > -1 \\ &= \frac{6}{(s+3)^4} + \frac{4s+4}{(s+1)^2+3^2} \quad s > -1\end{aligned}$$

Exercício 4.2.2.

a) $\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s\,(s+1)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{1}{s+1}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}\\
&= 1-e^{-t}
\end{aligned}$$

b) $\frac{s^2}{(s-1)^2} = \frac{2}{s-1} + \frac{1}{(s-1)^2} + 1$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s^2}{(s-1)^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{2}{s-1}+\frac{1}{(s-1)^2}+1\right\}\\
&= 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}+\mathcal{L}^{-1}\left\{1\right\}\\
&= 2e^t+\mathcal{L}^{-1}\left\{-\frac{d}{ds}\left(\frac{1}{s-1}\right)\right\}+\delta\,(t)\\
&= 2e^t+te^t+\delta\,(t)
\end{aligned}$$

c) $\frac{7}{(s+2)^2+3}$

$$\mathcal{L}^{-1}\left\{\frac{7}{(s+2)^2+3}\right\}=\frac{7\sqrt{3}}{3}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{(s+2)^2+3}\right\}=\frac{7\sqrt{3}e^{-2t}\sin\sqrt{3}t}{3}$$

d) $\frac{1}{s^2+2s+5} = \frac{1}{(s+1)^2+4}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}=\frac{e^{-t}\sin 2t}{2}$$

e) $\frac{-1}{(s-2)^2}$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{-1}{(s-2)^2}\right\} &= \mathcal{L}^{-1}\left\{-\frac{d}{ds}\left(\frac{-1}{s-2}\right)\right\}\\
&= -\mathcal{L}^{-1}\left\{-\frac{d}{ds}\left(\frac{1}{s-2}\right)\right\}\\
&= -te^{2t}
\end{aligned}$$

f) $\frac{s-7}{(s-7)^2+25}$

$$\mathcal{L}^{-1}\left\{\frac{s-7}{(s-7)^2+25}\right\}=e^{7t}\cos 5t$$

g) $\frac{s}{(s+a)^2+b^2} = \frac{s+a}{(s+a)^2+b^2} - \frac{a}{(s+a)^2+b^2}$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s}{(s+a)^2+b^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+b^2}\right\}-\mathcal{L}^{-1}\left\{\frac{a}{(s+a)^2+b^2}\right\}\\
&= e^{-at}\cos\,(bt)-\frac{a}{b}e^{-at}\sin\,(bt)
\end{aligned}$$

h) $\frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{-\frac{1}{a^2-b^2}}{s^2+a^2} + \frac{\frac{1}{a^2-b^2}}{s^2+b^2}$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} &= -\frac{1}{a^2-b^2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} + \frac{1}{a^2-b^2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+b^2}\right\} \\ &= -\frac{1}{a^2-b^2}\frac{\sin(at)}{a} + \frac{1}{a^2-b^2}\frac{\sin bt}{b}\end{aligned}$$

i) $\frac{s^2-5}{s^3+4s^2+3s} = \frac{1}{s(s+1)(s+3)} = \frac{1}{3s} - \frac{1}{2(s+1)} + \frac{1}{6(s+3)}$

$$\mathcal{L}^{-1}\left\{\frac{s^2-5}{s^3+4s^2+3s}\right\} = \frac{1}{3} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t}$$

Exercício 4.2.3.

a) $y'' + y = \sin t \quad y(0) = 1, y'(0) = 2$

$$\begin{aligned} & y'' + y = \sin t \\ \Leftrightarrow\ & \mathcal{L}\{y'' + y\} = \mathcal{L}\{\sin t\} \\ \Leftrightarrow\ & \mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\sin t\} \\ \Leftrightarrow\ & s^2Y - s - 2 + Y = \frac{1}{s^2+1} \qquad s > 0 \\ \Leftrightarrow\ & (s^2+1)Y = \frac{1}{s^2+1} + s + 2 \\ \Leftrightarrow\ & Y = \frac{1}{(s^2+1)^2} + \frac{s+2}{s^2+1} \\ \Leftrightarrow\ & \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2} + \frac{s+2}{s^2+1}\right\} \\ \Leftrightarrow\ & y = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ \Leftrightarrow\ & y = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} + \cos t + 2\sin t \end{aligned}$$

precisamos agora de cálcular $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$. Mais à frente veremos como calcular esta transformada inversa de Laplace usando convolução, mas por enquanto podemos calculá-la notando que

$$\begin{aligned}\frac{1}{(s^2+1)^2} &= \frac{s^2+1-s^2}{(s^2+1)^2} \\ &= \frac{1}{s^2+1} - \frac{s^2}{(s^2+1)^2} \\ &= \frac{1}{2}\frac{1}{s^2+1} + \frac{1}{2}\underbrace{\left(\frac{1}{s^2+1} - \frac{2s^2}{(s^2+1)^2}\right)}_{=\frac{d}{ds}\left(\frac{s}{s^2+1}\right)} \\ &= \frac{1}{2}\frac{1}{s^2+1} + \frac{1}{2}\frac{d}{ds}\left(\frac{s}{s^2+1}\right).\end{aligned}$$

Aplicando a transformada de Laplace inversa temos

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s^2+1}+\frac{1}{2}\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right\} \\
&= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}-\frac{1}{2}\mathcal{L}^{-1}\left\{-\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right\} \\
&= \frac{\sin t}{2}+\frac{1}{2}\mathcal{L}^{-1}\left\{\mathcal{L}\left\{t\cos t\right\}\right\} \\
&= \frac{\sin t}{2}-\frac{t\cos t}{2}
\end{aligned}$$

<u>Conclusão</u>:

$$\begin{aligned}
y &= \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}+\cos t+2\sin t \\
&= \frac{\sin t}{2}-\frac{t\cos t}{2}+\cos t+2\sin t \\
&= \frac{5}{2}\sin t+\frac{2-t}{2}\cos t
\end{aligned}$$

b) $y''+y=t\sin t \quad y(0)=1, y'(0)=2$

$$\begin{aligned}
& y''+y=t\sin t \\
\Leftrightarrow\ & \mathcal{L}\left\{y''+y\right\}=\mathcal{L}\left\{t\sin t\right\} \\
\Leftrightarrow\ & \mathcal{L}\left\{y''\right\}+\mathcal{L}\left\{y\right\}=-\frac{d}{ds}\left(\frac{1}{s^2+1}\right) \\
\Leftrightarrow\ & s^2Y-s-2+Y=\frac{2s}{(s^2+1)^2} \qquad s>0 \\
\Leftrightarrow\ & \left(s^2+1\right)Y=\frac{2s}{(s^2+1)^2}+s+2 \\
\Leftrightarrow\ & Y=\frac{2s}{(s^2+1)^3}+\frac{s+2}{s^2+1} \\
\Leftrightarrow\ & \mathcal{L}^{-1}\left\{Y\right\}=\mathcal{L}^{-1}\left\{\frac{2s}{(s^2+1)^3}+\frac{s+2}{s^2+1}\right\} \\
\Leftrightarrow\ & y=\mathcal{L}^{-1}\left\{\frac{2s}{(s^2+1)^3}\right\}+\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}+2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\
\Leftrightarrow\ & y=\mathcal{L}^{-1}\left\{\frac{2s}{(s^2+1)^3}\right\}+\cos t+2\sin t
\end{aligned}$$

Precisamos agora de calcular $\mathcal{L}^{-1}\left\{\frac{2s}{(s^2+1)^3}\right\}$. Note-se que

$$-\frac{d}{ds}\left(\frac{1}{2\left(s^2+1\right)^2}\right)=\frac{2s}{(s^2+1)^3}.$$

Aplicando a transformada de Laplace obtemos

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{2s}{(s^2+1)^3}\right\} &= \mathcal{L}^{-1}\left\{-\frac{d}{ds}\left(\frac{1}{2\left(s^2+1\right)^2}\right)\right\} \\
&= \frac{1}{2}\mathcal{L}^{-1}\left\{-\frac{d}{ds}\left(\frac{1}{(s^2+1)^2}\right)\right\}
\end{aligned}$$

mas pelo exercício anterior sabemos que

$$\begin{aligned} & \mathcal{L}^{-1}\left\{\frac{1}{\left(s^2+1\right)^2}\right\} = \frac{\sin t}{2} - \frac{t\cos t}{2} \\ \Leftrightarrow \quad & \mathcal{L}\left\{\frac{\sin t}{2} - \frac{t\cos t}{2}\right\} = \frac{1}{\left(s^2+1\right)^2} \end{aligned}$$

e portanto

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{2s}{\left(s^2+1\right)^3}\right\} &= \frac{1}{2}\mathcal{L}^{-1}\left\{-\frac{d}{ds}\left(\frac{1}{\left(s^2+1\right)^2}\right)\right\} \\ &= \frac{1}{2}\mathcal{L}^{-1}\left\{-\frac{d}{ds}\left(\mathcal{L}\left\{\frac{\sin t}{2} - \frac{t\cos t}{2}\right\}\right)\right\} \\ &= \frac{1}{2}\mathcal{L}^{-1}\left\{\mathcal{L}\left\{\frac{t\sin t}{2} - \frac{t^2\cos t}{2}\right\}\right\} \\ &= \frac{t\sin t}{4} - \frac{t^2\cos t}{4} \end{aligned}$$

<u>Conclusão</u>

$$\begin{aligned} y &= \mathcal{L}^{-1}\left\{\frac{2s}{\left(s^2+1\right)^3}\right\} + \cos t + 2\sin t \\ &= \frac{t\sin t}{4} - \frac{t^2\cos t}{4} + \cos t + 2\sin t \end{aligned}$$

c) $y'' - 2y' + 7y = \sin t \quad y(0) = 0, y'(0) = 0$

$$\begin{aligned} & y'' - 2y' + 7y = \sin t \\ \Leftrightarrow \quad & \mathcal{L}\left\{y'' - 2y' + 7y\right\} = \mathcal{L}\left\{\sin t\right\} \\ \Leftrightarrow \quad & \mathcal{L}\left\{y''\right\} - 2\mathcal{L}\left\{y'\right\} + 7\mathcal{L}\left\{y\right\} = \mathcal{L}\left\{\sin t\right\} \\ \Leftrightarrow \quad & s^2Y - 2sY + 7Y = \frac{1}{s^2+1} \\ \Leftrightarrow \quad & \left(s^2 - 2s + 7\right)Y = \frac{1}{s^2+1} \\ \Leftrightarrow \quad & Y = \frac{1}{\left(s^2+1\right)\left(s^2-2s+7\right)} \\ \Leftrightarrow \quad & Y = \frac{1}{20}\frac{s+3}{s^2+1} - \frac{1}{20}\frac{s+1}{s^2-2s+7} \\ \Leftrightarrow \quad & \mathcal{L}^{-1}\left\{Y\right\} = \mathcal{L}^{-1}\left\{\frac{1}{20}\frac{s+3}{s^2+1} - \frac{1}{20}\frac{s+1}{s^2-2s+7}\right\} \\ \Leftrightarrow \quad & y = \frac{1}{20}\mathcal{L}^{-1}\left\{\frac{s+3}{s^2+1}\right\} - \frac{1}{20}\mathcal{L}^{-1}\left\{\frac{s+1}{(s-1)^2+6}\right\} \\ \Leftrightarrow \quad & y = \frac{1}{20}\left(\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\right) - \frac{1}{20}\mathcal{L}^{-1}\left\{\frac{(s-1)+2}{(s-1)^2+6}\right\} \\ \Leftrightarrow \quad & y = \frac{\cos t + 3\sin t}{20} - \frac{1}{20}\left(\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+6}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+6}\right\}\right) \\ \Leftrightarrow \quad & y = \frac{\cos t + 3\sin t}{20} - \frac{1}{20}\left(e^t\mathcal{L}^{-1}\left\{\frac{s}{s^2+6}\right\} + 2e^t\mathcal{L}^{-1}\left\{\frac{1}{s^2+6}\right\}\right) \\ \Leftrightarrow \quad & y = \frac{\cos t + 3\sin t}{20} - \frac{1}{20}\left(e^t\cos\left(\sqrt{6}t\right) + \frac{2}{\sqrt{6}}e^t\sin\left(\sqrt{6}t\right)\right) \end{aligned}$$

d) $y'' - 2y' + y = te^t \quad y(0) = 0, y'(0) = 0$

$$\begin{aligned} y'' - 2y' + y = te^t &\Leftrightarrow \mathcal{L}\left\{y'' - 2y' + y\right\} = \mathcal{L}\left\{te^t\right\} \\ &\Leftrightarrow \mathcal{L}\left\{y''\right\} - 2\mathcal{L}\left\{y'\right\} + \mathcal{L}\left\{y\right\} = \mathcal{L}\left\{te^t\right\} \\ &\Leftrightarrow s^2 Y - 2sY + Y = -\frac{d}{ds}\left(\frac{1}{s-1}\right) \\ &\Leftrightarrow (s-1)^2 Y = \frac{1}{(s-1)^2} \\ &\Leftrightarrow Y = \frac{1}{(s-1)^4} \\ &\Leftrightarrow \mathcal{L}^{-1}\left\{Y\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^4}\right\} \\ &\Leftrightarrow y = \frac{1}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{(s-1)^4}\right\} \\ &\Leftrightarrow y = \frac{t^3 e^t}{6} \end{aligned}$$

e) $y'' + y' + y = 1 + e^{-t} \quad y(0) = 3, y'(0) = -5$

$$\begin{aligned} & y'' + y' + y = 1 + e^{-t} \\ \Leftrightarrow\ & \mathcal{L}\left\{y'' + y' + y\right\} = \mathcal{L}\left\{1 + e^{-t}\right\} \\ \Leftrightarrow\ & \mathcal{L}\left\{y''\right\} + \mathcal{L}\left\{y'\right\} + \mathcal{L}\left\{y\right\} = \mathcal{L}\left\{1\right\} + \mathcal{L}\left\{e^{-t}\right\} \\ \Leftrightarrow\ & s^2 Y - 3s + 5 + sY - 3 + Y = \frac{1}{s} + \frac{1}{s+1} \\ \Leftrightarrow\ & \left(s^2 + s + 1\right) Y = \frac{1}{s} + \frac{1}{s+1} + 3s - 2 \\ \Leftrightarrow\ & Y = \frac{s^2 + 3s^3 + 1}{s(s+1)(s^2+s+1)} \\ \Leftrightarrow\ & Y = \frac{1}{s} + \frac{1}{s+1} + \frac{s-3}{s^2+s+1} \\ \Leftrightarrow\ & \mathcal{L}^{-1}\left\{Y\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{1}{s+1} + \frac{s-3}{s^2+s+1}\right\} \\ \Leftrightarrow\ & y = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{s-3}{s^2+s+1}\right\} \\ \Leftrightarrow\ & y = 1 + e^{-t} + \mathcal{L}^{-1}\left\{\frac{s + \frac{1}{2} - \frac{7}{2}}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}}\right\} \\ \Leftrightarrow\ & y = 1 + e^{-t} + \mathcal{L}^{-1}\left\{\frac{s + \frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}}\right\} - \frac{7\sqrt{3}}{3}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}}\right\} \\ \Leftrightarrow\ & y = 1 + e^{-t} + e^{-\frac{t}{2}}\mathcal{L}^{-1}\left\{\frac{s}{s^2 + \frac{3}{4}}\right\} - \frac{7\sqrt{3}}{3}e^{-\frac{t}{2}}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{3}}{2}}{s^2 + \frac{3}{4}}\right\} \\ \Leftrightarrow\ & y = 1 + e^{-t} + e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{7\sqrt{3}}{3}e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right) \end{aligned}$$

Exercício 4.2.4. Neste exercício vamos usar que

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{+\infty} F(u)\,du, \quad \text{onde} \quad F(u) = \mathcal{L}\left\{f(t)\right\}$$

a) $\frac{\sin t}{t}$

$$\begin{aligned}\mathcal{L}\left\{\frac{\sin t}{t}\right\}=\int_s^{+\infty}\frac{1}{u^2+1}du &= \lim_{r\to+\infty}\int_s^r\frac{1}{u^2+1}du\\ &= \lim_{r\to+\infty} arctg\,(u)|_s^r\\ &= \lim_{r\to+\infty}\left(arctg\,(r)-arctg\,(s)\right)\\ &= \frac{\pi}{2}-arctg\,(s)\end{aligned}$$

b) $\frac{\cos(at)-1}{t}$. Ora

$$F\,(s)=\mathcal{L}\left\{\cos\,(at)-1\right\}=\frac{s}{s^2+a^2}-\frac{1}{s}$$

e portanto

$$\begin{aligned}\mathcal{L}\left\{\frac{\cos\,(at)-1}{t}\right\} &= \int_s^{+\infty}\left(\frac{u}{u^2+a^2}-\frac{1}{u}\right)du\\ &= \lim_{r\to+\infty}\int_s^r\left(\frac{u}{u^2+a^2}-\frac{1}{u}\right)du\\ &= \lim_{r\to+\infty}\left(\frac{\ln\left(u^2+a^2\right)}{2}-\ln u\right)\Bigg|_s^r\\ &= \frac{1}{2}\lim_{r\to+\infty}\left(\ln\left(u^2+a^2\right)-2\ln u\right)\Big|_s^r\\ &= \frac{1}{2}\lim_{r\to+\infty}\left(\ln\left(u^2+a^2\right)-\ln u^2\right)\Big|_s^r\\ &= \frac{1}{2}\lim_{r\to+\infty}\ln\left(1+\frac{a^2}{u^2}\right)\Bigg|_s^r\\ &= \frac{1}{2}\lim_{r\to+\infty}\left(\ln\left(1+\frac{a^2}{r^2}\right)-\ln\left(1+\frac{a^2}{s^2}\right)\right)\\ &= -\frac{1}{2}\ln\left(1+\frac{a^2}{s^2}\right)\end{aligned}$$

c) $\frac{e^{at}-e^{bt}}{t}$. Ora

$$F\,(s)=\mathcal{L}\left\{e^{at}-e^{bt}\right\}=\frac{1}{s-a}-\frac{1}{s-b}$$

e portanto

$$\begin{aligned}\mathcal{L}\left\{\frac{e^{at}-e^{bt}}{t}\right\} &= \int_s^{+\infty}\left(\frac{1}{u-a}-\frac{1}{u-b}\right)du\\ &= \lim_{r\to+\infty}\int_s^r\left(\frac{1}{u-a}-\frac{1}{u-b}\right)du\\ &= \lim_{r\to+\infty}\left(\ln|u-a|-\ln|u-b|\right)|_s^r\\ &= \lim_{r\to+\infty}\ln\left|\frac{u-a}{u-b}\right|\Bigg|_s^r\\ &= \lim_{r\to+\infty}\left(\ln\left|\frac{r-a}{r-b}\right|-\ln\left|\frac{s-a}{s-b}\right|\right)\\ &= \lim_{r\to+\infty}\left(\ln\left|1+\frac{b-a}{r-b}\right|-\ln\left|\frac{s-a}{s-b}\right|\right)\\ &= -\ln\left|\frac{s-a}{s-b}\right|\end{aligned}$$

Exercício 4.2.5. Neste exercício vamos usar que

$$F(s) = \mathcal{L}\{f(t)\} \Longrightarrow f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}$$

a) $\ln\left(\frac{s+a}{s-a}\right)$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\ln\left(\frac{s+a}{s-a}\right)\right\} &= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{2a}{(s-a)(s+a)}\right\} \\
&= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} \\
&= -\frac{1}{t}\left(e^{at} - e^{-at}\right) \\
&= -\frac{2\sinh(at)}{t}
\end{aligned}$$

b) $\ln\left(1 - \frac{a^2}{s^2}\right)$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\ln\left(1 - \frac{a^2}{s^2}\right)\right\} &= -\frac{1}{t}\mathcal{L}^{-1}\left\{-\frac{2a^2}{s(s^2-a^2)}\right\} \\
&= \frac{1}{t}\mathcal{L}^{-1}\left\{-\frac{2}{s} + \frac{1}{s+a} + \frac{1}{s-a}\right\} \\
&= \frac{1}{t}\left(-2 + e^{-at} + e^{at}\right) \\
&= 2\frac{\cosh at - 1}{t}
\end{aligned}$$

c) $arctg\left(\frac{a}{s}\right)$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{arctg\left(\frac{a}{s}\right)\right\} &= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{-\frac{a}{s^2}}{1+\left(\frac{a}{s}\right)^2}\right\} \\
&= -\frac{1}{t}\mathcal{L}^{-1}\left\{-\frac{a}{s^2+a^2}\right\} \\
&= \frac{\sin at}{t}
\end{aligned}$$

1 4.3. Função Degrau Unitário de Heaviside

Exercício 4.3.1.

a) $\frac{e^{-s}}{s}$

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\right\} = H_1(t)$$

b) $\frac{e^{-s}}{s^2}$

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2}\right\} = \mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s^2}\right\} = \mathcal{L}^{-1}\left\{e^{-s}\mathcal{L}\{t\}\right\} = (t-1)H_1(t)$$

c) $\frac{e^{-3s}}{s^2+4}$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2+4}\right\}=\mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{s^2+4}\right\} &= \frac{1}{2}\mathcal{L}^{-1}\left\{e^{-3s}\mathcal{L}\left\{\sin(2t)\right\}\right\}\\ &= \frac{1}{2}H_3(t)\sin(2t-6)\end{aligned}$$

d) $\frac{e^{-s}-2e^{-2s}+2e^{-3s}-4e^{-4s}}{s^2}$

$$\begin{aligned}&\mathcal{L}^{-1}\left\{\frac{e^{-s}-2e^{-2s}+2e^{-3s}-4e^{-4s}}{s^2}\right\}=\\ &= \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2}\right\}-2\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}+2\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\}-4\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s^2}\right\}\\ &= (t-1)H_1(t)-2(t-2)H_2(t)+2(t-3)H_3(t)-4(t-4)H_4(t)\end{aligned}$$

e) $\frac{e^{-3s}}{s^2-2s-3}=\frac{e^{-3s}}{(s-1)^2-2^2}$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s-1)^2-2^2}\right\} &= \mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{(s-1)^2-2^2}\right\}\\ &= \frac{1}{2}\mathcal{L}^{-1}\left\{e^{-3s}\mathcal{L}\left\{e^t\sinh 2t\right\}\right\}\\ &= \frac{e^{t-3}\sinh(2t-6)}{2}H_3(t)\end{aligned}$$

Exercício 4.3.2.

a) $y''+4y=\begin{cases}1 & , \quad 0\le t<4\\ 0 & , \quad t>4\end{cases}$ $\quad y(0)=3, y'(0)=-2$. Note-se que

$$H_0(t)-H_4(t)=\begin{cases}1 & , \quad 0\le t<4\\ 0 & , \quad t>4\end{cases}$$

e portanto

$$\begin{aligned}& y''+4y=H_0(t)-H_4(t)\\ \Leftrightarrow\ & \mathcal{L}\left\{y''+4y\right\}=\mathcal{L}\left\{H_0(t)-H_4(t)\right\}\\ \Leftrightarrow\ & \mathcal{L}\left\{y''\right\}+4\mathcal{L}\left\{y\right\}=\mathcal{L}\left\{H_0(t)\right\}-\mathcal{L}\left\{H_4(t)\right\}\\ \Leftrightarrow\ & s^2Y-3s+2+4Y=\frac{1}{s}-\frac{e^{-4s}}{s}\\ \Leftrightarrow\ & \left(s^2+4\right)Y=\frac{1}{s}-\frac{e^{-4s}}{s}+3s-2\\ \Leftrightarrow\ & Y=\frac{1}{s(s^2+4)}-\frac{e^{-4s}}{s(s^2+4)}+\frac{3s-2}{s^2+4}\\ \Leftrightarrow\ & Y=\frac{1}{s(s^2+4)}-\frac{e^{-4s}}{s(s^2+4)}+\frac{3s-2}{s^2+4}\\ \Leftrightarrow\ & \mathcal{L}^{-1}\left\{Y\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}-\frac{e^{-4s}}{s(s^2+4)}+\frac{3s-2}{s^2+4}\right\}\\ \Leftrightarrow\ & y=\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\}-\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s(s^2+4)}\right\}+3\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}-\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}\\ \Leftrightarrow\ & y=\mathcal{L}^{-1}\left\{\frac{1}{4s}-\frac{1}{4}\frac{s}{s^2+4}\right\}-\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s(s^2+4)}\right\}+3\cos(2t)-\sin(2t)\end{aligned}$$

$$\Leftrightarrow \quad y = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s\left(s^2+4\right)}\right\} + 3\cos(2t) - \sin(2t)$$

$$\Leftrightarrow \quad y = \frac{1-\cos(2t)}{4} - \mathcal{L}^{-1}\left\{e^{-4s}\mathcal{L}\left\{\frac{1-\cos(2t)}{4}\right\}\right\} + 3\cos(2t) - \sin(2t)$$

$$\Leftrightarrow \quad y = \frac{1+11\cos(2t)}{4} - H_4(t)\frac{1-\cos(2t-8)}{4} - \sin(2t)$$

b) $y'' + y = \begin{cases} \cos t & , \quad 0 \leq t < \frac{\pi}{2} \\ 0 & , \quad t \geq \frac{\pi}{2} \end{cases}$ $\quad y(0) = 3, y'(0) = -1$. Note-se que

$$\left(H_0(t) - H_{\frac{\pi}{2}}(t)\right)\cos t = \begin{cases} \cos t & , \quad 0 \leq t < \frac{\pi}{2} \\ 0 & , \quad t \geq \frac{\pi}{2} \end{cases}$$

e portanto

$$y'' + y = \left(H_0(t) - H_{\frac{\pi}{2}}(t)\right)\cos t$$

$$\Leftrightarrow \quad \mathcal{L}\left\{y'' + y\right\} = \mathcal{L}\left\{H_0(t)\cos t - H_{\frac{\pi}{2}}(t)\cos t\right\}$$

$$\Leftrightarrow \quad \mathcal{L}\left\{y''\right\} + \mathcal{L}\{y\} = \mathcal{L}\{H_0(t)\cos t\} - \mathcal{L}\left\{H_{\frac{\pi}{2}}(t)\cos t\right\}$$

$$\Leftrightarrow \quad \mathcal{L}\left\{y''\right\} + \mathcal{L}\{y\} = \mathcal{L}\{H_0(t)\cos t\} - \mathcal{L}\left\{H_{\frac{\pi}{2}}(t)\sin\left(t - \frac{\pi}{2}\right)\right\}$$

$$\Leftrightarrow \quad s^2Y - 3s + 1 + Y = \mathcal{L}\{\cos t\} - e^{-\frac{\pi}{2}s}\mathcal{L}\{\sin t\}$$

$$\Leftrightarrow \quad Y = \frac{s}{\left(s^2+1\right)^2} - \frac{e^{-\frac{\pi}{2}s}}{\left(s^2+1\right)^2} - \frac{3s+1}{s^2+1}$$

$$\Leftrightarrow \quad \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{s}{\left(s^2+1\right)^2} - \frac{e^{-\frac{\pi}{2}s}}{\left(s^2+1\right)^2} - \frac{3s+1}{s^2+1}\right\}$$

$$\Leftrightarrow \quad y = \mathcal{L}^{-1}\left\{\frac{s}{\left(s^2+1\right)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-\frac{\pi}{2}s}}{\left(s^2+1\right)^2}\right\} + 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\Leftrightarrow \quad y = \frac{1}{2}\mathcal{L}^{-1}\left\{-\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\} - \mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\frac{1}{\left(s^2+1\right)^2}\right\} + 3\cos(t) - \sin(t)$$

no exercício 4.2.3.a) vimos que

$$\mathcal{L}^{-1}\left\{\frac{1}{\left(s^2+1\right)^2}\right\} = \frac{\sin t}{2} - \frac{t\cos t}{2}$$

$$\Leftrightarrow \quad \frac{1}{\left(s^2+1\right)^2} = \mathcal{L}\left\{\frac{\sin t}{2} - \frac{t\cos t}{2}\right\}$$

e portanto

$$y = \frac{1}{2}\mathcal{L}^{-1}\left\{-\frac{d}{ds}\mathcal{L}\{\sin t\}\right\} - \mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\mathcal{L}\left\{\frac{\sin t - t\cos t}{2}\right\}\right\} + 3\cos(t) - \sin(t)$$

$$\Leftrightarrow \quad y = \frac{t\sin t}{2} - H_{\frac{\pi}{2}}(t)\frac{\sin\left(t-\frac{\pi}{2}\right) - \left(t-\frac{\pi}{2}\right)\cos\left(t-\frac{\pi}{2}\right)}{2} + 3\cos(t) - \sin(t)$$

$$\Leftrightarrow \quad y = \frac{t\sin t}{2} + H_{\frac{\pi}{2}}(t)\frac{\cos t + \left(t-\frac{\pi}{2}\right)\sin t}{2} + 3\cos(t) - \sin(t)$$

c) $y'' + y' + 7y = \begin{cases} t & , \quad 0 \leq t < 2 \\ 0 & , \quad t \geq 2 \end{cases}$ $\quad y(0) = 0, y'(0) = 0$. Note-se que

$$\left(H_0(t) - H_2(t)\right)t = \begin{cases} t & , \quad 0 \leq t < 2 \\ 0 & , \quad t \geq 2 \end{cases}$$

e portanto

$$\begin{aligned}
& y''+y'+7y=tH_0(t)-tH_2(t)\\
\Leftrightarrow\ & \mathcal{L}\left\{y''+y'+7y\right\}=\mathcal{L}\left\{tH_0(t)-tH_2(t)\right\}\\
\Leftrightarrow\ & \mathcal{L}\left\{y''\right\}+\mathcal{L}\left\{y'\right\}+7\mathcal{L}\left\{y\right\}=\mathcal{L}\left\{tH_0(t)\right\}-\mathcal{L}\left\{tH_2(t)\right\}\\
\Leftrightarrow\ & s^2Y+sY+7Y=-\frac{d}{ds}\mathcal{L}\left\{H_0(t)\right\}+\frac{d}{ds}\mathcal{L}\left\{H_2(t)\right\}\\
\Leftrightarrow\ & \left(s^2+s+7\right)Y=-\frac{d}{ds}\left(\frac{1}{s}\right)+\frac{d}{ds}\left(\frac{e^{-2s}}{s}\right)\\
\Leftrightarrow\ & \left(s^2+s+7\right)Y=\frac{1}{s^2}-\frac{e^{-2s}(1+2s)}{s^2}\\
\Leftrightarrow\ & Y=\frac{1}{s^2\left(s^2+s+7\right)}-\frac{e^{-2s}(1+2s)}{s^2\left(s^2+s+7\right)}\\
\Leftrightarrow\ & \mathcal{L}^{-1}\left\{Y\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s^2\left(s^2+s+7\right)}\right\}-\mathcal{L}^{-1}\left\{\frac{e^{-2s}(1+2s)}{s^2\left(s^2+s+7\right)}\right\}
\end{aligned}$$

mas

$$\begin{aligned}
& \mathcal{L}^{-1}\left\{\frac{1}{s^2\left(s^2+s+7\right)}\right\}\\
=\ & \mathcal{L}^{-1}\left\{-\frac{1}{49s}+\frac{1}{7s^2}+\frac{1}{49}\frac{s-6}{s^2+s+7}\right\}\\
=\ & -\frac{1}{49}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+\frac{1}{7}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}+\frac{1}{49}\mathcal{L}^{-1}\left\{\frac{\left(s+\frac{1}{2}\right)-\frac{13}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{27}{4}}\right\}\\
=\ & -\frac{1}{49}+\frac{t}{7}+\frac{1}{49}\left(\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{27}{4}}\right\}-\frac{13}{\sqrt{27}}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{27}}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{27}{4}}\right\}\right)\\
=\ & -\frac{1}{49}+\frac{t}{7}+\frac{1}{49}\left(e^{-\frac{t}{2}}\mathcal{L}^{-1}\left\{\frac{s}{s^2+\frac{27}{4}}\right\}-\frac{13}{\sqrt{27}}e^{-\frac{t}{2}}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{27}}{2}}{s^2+\frac{27}{4}}\right\}\right)\\
=\ & -\frac{1}{49}+\frac{t}{7}+\frac{1}{49}\left(e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{27}}{2}t\right)-\frac{13}{\sqrt{27}}e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{27}}{2}t\right)\right)
\end{aligned}$$

e

$$\begin{aligned}
\mathcal{L}^{-1}\left\{e^{-2s}\frac{1+2s}{s^2\left(s^2+s+7\right)}\right\} &= \mathcal{L}^{-1}\left\{e^{-2s}\left(\frac{13}{49s}+\frac{1}{7s^2}-\frac{1}{49}\frac{13s+20}{s^2+s+7}\right)\right\}\\
&= H_2(t)\,f(t-2)
\end{aligned}$$

onde

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}\left\{\frac{13}{49s}+\frac{1}{7s^2}-\frac{1}{49}\frac{13s+20}{s^2+s+7}\right\}\\
&= \frac{13}{49}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+\frac{1}{7}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}-\frac{1}{49}\mathcal{L}^{-1}\left\{\frac{13s+20}{s^2+s+7}\right\}\\
&= \frac{13}{49}+\frac{t}{7}-\frac{1}{49}\mathcal{L}^{-1}\left\{\frac{13s+20}{\left(s+\frac{1}{2}\right)^2+\frac{27}{4}}\right\}\\
&= \frac{13}{49}+\frac{t}{7}-\frac{1}{49}\mathcal{L}^{-1}\left\{\frac{13\left(s+\frac{1}{2}\right)+\frac{27}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{27}{4}}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{13}{49}+\frac{t}{7}-\frac{1}{49}\left(13\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{27}{4}}\right\}+\sqrt{27}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{27}}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{27}{4}}\right\}\right)\\
&= \frac{13}{49}+\frac{t}{7}-\frac{1}{49}\left(13e^{-\frac{t}{2}}\mathcal{L}^{-1}\left\{\frac{s}{s^2+\frac{27}{4}}\right\}+\sqrt{27}e^{-\frac{t}{2}}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{27}}{2}}{s^2+\frac{27}{4}}\right\}\right)\\
&= \frac{13}{49}+\frac{t}{7}-\frac{1}{49}\left(13e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{27}}{2}t\right)+\sqrt{27}e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{27}}{2}t\right)\right).
\end{aligned}
$$

Retomando a resolução da equação diferencial temos que

$$
\begin{aligned}
y &= -\frac{1}{49}+\frac{t}{7}+\frac{1}{49}\left(e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{27}}{2}t\right)-\frac{13}{\sqrt{27}}e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{27}}{2}t\right)\right)-H_2(t)\left(\frac{13}{49}+\frac{t-2}{7}\right)\\
&\quad +\frac{1}{49}H_2(t)\left(13e^{-\frac{t-2}{2}}\cos\left(\frac{\sqrt{27}}{2}t-\sqrt{27}\right)+\sqrt{27}e^{-\frac{t-2}{2}}\sin\left(\frac{\sqrt{27}}{2}t-\sqrt{27}\right)\right)
\end{aligned}
$$

4.4. Função δ de Dirac

Exercício 4.4.1.

a) $y''-4y'+4y=3\delta(t-1)+\delta(t-2), \quad y(0)=1, y'(0)=1.$

$$
\begin{aligned}
& y''-4y'+4y=3\delta(t-1)+\delta(t-2)\\
\Leftrightarrow\ & \mathcal{L}\left\{y''-4y'+4y\right\}=\mathcal{L}\left\{3\delta(t-1)+\delta(t-2)\right\}\\
\Leftrightarrow\ & s^2Y-s-1-4(sY-1)+4Y=3e^{-s}+e^{-2s}\\
\Leftrightarrow\ & \left(s^2-4s-4\right)Y=3e^{-t}+e^{-2s}+s-3\\
\Leftrightarrow\ & Y=\frac{3e^{-s}}{(s-2)^2}+\frac{e^{-2s}}{(s-2)^2}+\frac{s-3}{(s-2)^2}\\
\Leftrightarrow\ & \mathcal{L}^{-1}\{Y\}=\mathcal{L}^{-1}\left\{\frac{3e^{-s}}{(s-2)^2}+\frac{e^{-2s}}{(s-2)^2}+\frac{s-3}{(s-2)^2}\right\}\\
\Leftrightarrow\ & y=3\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{(s-2)^2}\right\}+\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{(s-2)^2}\right\}+\mathcal{L}^{-1}\left\{\frac{s-3}{(s-2)^2}\right\}\\
\Leftrightarrow\ & y=3\mathcal{L}^{-1}\left\{e^{-s}\mathcal{L}\left\{te^{2t}\right\}\right\}+\mathcal{L}^{-1}\left\{e^{-2s}\mathcal{L}\left\{te^{2t}\right\}\right\}+\mathcal{L}^{-1}\left\{\frac{(s-2)-1}{(s-2)^2}\right\}\\
\Leftrightarrow\ & y=3(t-1)e^{2t-2}H_1(t)+(t-2)e^{2t-4}H_2(t)+\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\}\\
\Leftrightarrow\ & y=3(t-1)e^{2t-2}H_1(t)+(t-2)e^{2t-4}H_2(t)+e^{2t}-te^{2t}
\end{aligned}
$$

b) $y''+y=\sin t+\delta(t-\pi), \quad y(0)=0, y'(0)=0$

$$
\begin{aligned}
& y''+y=\sin t+\delta(t-\pi)\\
\Leftrightarrow\ & \mathcal{L}\left\{y''+y\right\}=\mathcal{L}\left\{\sin t+\delta(t-\pi)\right\}\\
\Leftrightarrow\ & \left(s^2+1\right)Y=\frac{1}{s^2+1}+e^{-\pi t}\\
\Leftrightarrow\ & Y=\frac{1}{\left(s^2+1\right)^2}+\frac{e^{-\pi t}}{s^2+1}
\end{aligned}
$$

$$\begin{aligned}
&\Leftrightarrow\ \mathcal{L}^{-1}\{Y\}=\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}+\mathcal{L}^{-1}\left\{\frac{e^{-\pi t}}{s^2+1}\right\}\\
&\Leftrightarrow\ y=\frac{\sin t}{2}-\frac{t\cos t}{2}+\mathcal{L}^{-1}\left\{e^{-\pi t}\mathcal{L}\{\sin t\}\right\}\\
&\Leftrightarrow\ y=\frac{\sin t}{2}-\frac{t\cos t}{2}+\sin(t-\pi)\,H_\pi(t)\\
&\Leftrightarrow\ y=\frac{\sin t}{2}-\frac{t\cos t}{2}-\sin(t)\,H_\pi(t)
\end{aligned}$$

c) $y''+2y'+y=e^{-t}+3\delta(t-3),\quad y(0)=0, y'(0)=3$

$$\begin{aligned}
&y''+2y'+y=e^{-t}+3\delta(t-3)\\
\Leftrightarrow\ &\mathcal{L}\left\{y''+2y'+y\right\}=\mathcal{L}\left\{e^{-t}+3\delta(t-3)\right\}\\
\Leftrightarrow\ &s^2Y-3+2sY+Y=\frac{1}{s+1}+3e^{-3s}\\
\Leftrightarrow\ &(s+1)^2\,Y=\frac{1}{s+1}+3e^{-3s}+3\\
\Leftrightarrow\ &Y=\frac{1}{(s+1)^3}+3\frac{e^{-3s}}{(s+1)^2}+\frac{3}{(s+1)^2}\\
\Leftrightarrow\ &\mathcal{L}^{-1}\{Y\}=\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}+3\frac{e^{-3s}}{(s+1)^2}+\frac{3}{(s+1)^2}\right\}\\
\Leftrightarrow\ &y=\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\}+3\mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{(s+1)^2}\right\}+3\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}\\
\Leftrightarrow\ &y=\mathcal{L}^{-1}\left\{\mathcal{L}\left\{\frac{t^2e^{-t}}{2}\right\}\right\}+3\mathcal{L}^{-1}\left\{e^{-3s}\mathcal{L}\left\{te^{-t}\right\}\right\}+3\mathcal{L}^{-1}\left\{\mathcal{L}\left\{te^{-t}\right\}\right\}\\
\Leftrightarrow\ &y=\frac{t^2e^{-t}}{2}+3(t-3)\,e^{-t+3}H_3(t)+3te^{-t}
\end{aligned}$$

d) $y'+y=\delta(t)+\delta(t-1),\quad y(0)=0$

$$\begin{aligned}
&y'+y=\delta(t)+\delta(t-1)\\
\Leftrightarrow\ &\mathcal{L}\left\{y'+y\right\}=\mathcal{L}\{\delta(t)+\delta(t-1)\}\\
\Leftrightarrow\ &(s+1)\,Y=1+e^{-s}\\
\Leftrightarrow\ &Y=\frac{1}{s+1}+\frac{e^{-s}}{s+1}\\
\Leftrightarrow\ &\mathcal{L}^{-1}\{Y\}=\mathcal{L}^{-1}\left\{\frac{1}{s+1}+\frac{e^{-s}}{s+1}\right\}\\
\Leftrightarrow\ &y=\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}+\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s+1}\right\}\\
\Leftrightarrow\ &y=e^{-t}+\mathcal{L}^{-1}\left\{e^{-s}\mathcal{L}\left\{e^{-t}\right\}\right\}\\
\Leftrightarrow\ &y=e^{-t}+e^{1-t}H_1(t)
\end{aligned}$$

e) $y''+y=\delta(t-\pi)-\delta(t-2\pi),\quad y(0)=0, y'(0)=2$

$$\begin{aligned}
&y''+y=\delta(t-\pi)-\delta(t-2\pi)\\
\Leftrightarrow\ &\mathcal{L}\left\{y''+y\right\}=\mathcal{L}\{\delta(t-\pi)-\delta(t-2\pi)\}\\
\Leftrightarrow\ &\mathcal{L}\left\{y''\right\}+\mathcal{L}\{y\}=\mathcal{L}\{\delta(t-\pi)\}-\mathcal{L}\{\delta(t-2\pi)\}\\
\Leftrightarrow\ &s^2Y-2+Y=e^{-\pi s}-e^{-2\pi s}\\
\Leftrightarrow\ &Y=\frac{2}{s^2+1}+\frac{e^{-\pi s}}{s^2+1}-\frac{e^{-2\pi s}}{s^2+1}
\end{aligned}$$

$$
\begin{aligned}
&\Leftrightarrow \quad \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{2}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1}\right\} \\
&\Leftrightarrow \quad y = 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+1}\right\} \\
&\Leftrightarrow \quad y = 2\sin t + H_\pi(t)\sin(t-\pi) - H_{2\pi}(t)\sin(t-2\pi) \\
&\Leftrightarrow \quad y = 2\sin t - H_\pi(t)\sin t - H_{2\pi}(t)\sin(t)
\end{aligned}
$$

4.5. Convolução

Exercício 4.5.1.

a) $f(t) = e^{at}, \quad g(t) = e^{bt}, \quad a \neq b$

$$
f * g = \int_0^t e^{a(t-u)} e^{bu} du = \int_0^t e^{at-au} e^{bu} du = \frac{e^{at}}{b-a}\left[e^{(b-a)u}\right]_0^t = \frac{e^{bt}-e^{at}}{b-a}
$$

b) $f(t) = \cos(at), \quad g(t) = \sin(bt)$

$$
\begin{aligned}
f * g &= \int_0^t \cos(at-au)\sin(bu)\, du \\
&= \int_0^t \big(\cos(at)\cos(au)\sin(bu) + \sin(at)\sin^2(bu)\big)\, du \\
&= \cos(at)\int_0^t \cos(au)\sin(bu)\, du + \sin(at)\int_0^t \sin^2(bu)\, du \\
&= \cos(at)\int_0^t \frac{\sin(bu+au)+\sin(bu-au)}{2} du + \sin(at)\int_0^t \frac{1-\cos(2bu)}{2} du \\
&= \frac{\cos(at)}{2}\left[-\frac{\cos(bu+au)}{a+b} + \frac{\cos(bu-au)}{a-b}\right]_0^t + \frac{\sin(at)}{2}\left[u - \frac{\sin(2bu)}{2b}\right]_0^t \\
&= \frac{\cos(at)}{2}\left[-\frac{\cos(bt+at)}{a+b} + \frac{\cos(bt-at)}{a-b} - \frac{2b}{a^2-b^2}\right] + \frac{\sin(at)}{2}\left[t - \frac{\sin(2bt)}{2b}\right]
\end{aligned}
$$

c) $f(t) = t, \quad g(t) = \sin(t)$

$$
\begin{aligned}
f * g = \int_0^t (t-u)\sin(u)\, du &= t\int_0^t \sin(u)\, du - \int_0^t u\sin(u)\, du \\
&= -t\cos u|_0^t - \int_0^t u\sin(u)\, du \\
&= -t\cos t + t - \left(-u\cos u|_0^t + \int_0^t \cos u du\right) \\
&= -t\cos t + t + t\cos t - \sin t|_0^t \\
&= t - \sin t
\end{aligned}
$$

Exercício 4.5.2.

a) $\frac{1}{s(s-1)}$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s\,(s-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\frac{1}{s-1}\right\} &= \mathcal{L}^{-1}\left\{\mathcal{L}\left\{1\right\}\mathcal{L}\left\{e^t\right\}\right\}\\ &= 1 * e^t\\ &= \int_0^t (t-u)\,e^u du\\ &= e^t - t - 1\end{aligned}$$

b) $\frac{1}{s^2(s^2+1)}$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2\,(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\frac{1}{s^2+1}\right\} &= \mathcal{L}^{-1}\left\{\mathcal{L}\left\{t\right\}\mathcal{L}\left\{\sin t\right\}\right\}\\ &= t * \sin t\\ &= \int_0^t (t-u)\sin(u)\,du\\ &= t - \sin t\end{aligned}$$

c) $\frac{4}{s^2(s-2)}$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{4}{s^2\,(s-2)}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{s^2}\frac{1}{s-2}\right\} &= \mathcal{L}^{-1}\left\{\mathcal{L}\left\{4t\right\}\mathcal{L}\left\{e^{2t}\right\}\right\}\\ &= (4t) * e^{2t}\\ &= \int_0^t 4\,(t-u)\,e^{2u} du\\ &= e^{2t} - 2t - 1\end{aligned}$$

d) $\frac{s}{(s^2+4)(s+1)}$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)\,(s+1)}\right\} &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\frac{1}{s+1}\right\}\\ &= \mathcal{L}^{-1}\left\{\mathcal{L}\left\{\cos(2t)\right\}\mathcal{L}\left\{e^{-t}\right\}\right\}\\ &= \cos(2t) * e^{-t}\\ &= \int_0^t \cos(2t-2u)\,e^{-u} du\\ &= \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t - \frac{1}{5}e^{-t}\end{aligned}$$

e) $\frac{s}{(s^2+1)^2}$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\frac{1}{s^2+1}\right\}\\ &= \mathcal{L}^{-1}\left\{\mathcal{L}\left\{\cos t\right\}\mathcal{L}\left\{\sin t\right\}\right\}\\ &= \cos t * \sin t\\ &= \int_0^t \cos(t-u)\sin u du\\ &= \int_0^t (\cos t\cos u + \sin t\sin u)\sin u du\\ &= \cos t\int_0^t \cos u\sin u du + \sin t\int_0^t \sin^2 u du\end{aligned}$$

$$
\begin{aligned}
&= \left.\frac{\cos t \sin^2 u}{2}\right|_0^t + \frac{\sin t}{2}\int_0^t (1-\cos 2u)\, du \\
&= \frac{\cos t \sin^2 t}{2} + \frac{\sin t}{2}\left[u - \frac{\sin 2u}{2}\right]_0^t \\
&= \frac{\cos t \sin^2 t}{2} + \frac{\sin t}{2}\left(t - \frac{\sin 2t}{2}\right) \\
&= \frac{\cos t \sin^2 t}{2} + \frac{\sin t}{2}\left(t - \sin t \cos t\right) \\
&= \frac{1}{2} t \sin t
\end{aligned}
$$

Calculemos agora algumas transformadas inversas que ficaram para trás

- $\frac{1}{\left(s^2+1\right)^2}$

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{\left(s^2+1\right)^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\frac{1}{s^2+1}\right\} \\
&= \mathcal{L}^{-1}\left\{\mathcal{L}\left\{\sin t\right\}\mathcal{L}\left\{\sin t\right\}\right\} \\
&= \sin t * \sin t \\
&= \int_0^t \sin\left(t-u\right)\sin u du \\
&= \int_0^t \sin t \cos u \sin u du - \int_0^t \cos t \sin^2 u du \\
&= \left.\frac{\sin t \sin^2 u}{2}\right|_0^t - \frac{\cos t}{2}\left[u - \frac{\sin 2u}{2}\right]_0^t \\
&= \frac{\sin t \sin^2 t}{2} - \frac{\cos t}{2}\left(t - \sin t \cos t\right) \\
&= \frac{\sin t\left(1-\cos^2 t\right)}{2} - \frac{t\cos t}{2} + \frac{\sin t \cos^2 t}{2} \\
&= \frac{\sin t}{2} - \frac{t \cos t}{2}
\end{aligned}
$$

- $\frac{1}{\left(s^2+1\right)^3}$

$$
\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{1}{\left(s^2+1\right)^3}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\frac{1}{\left(s^2+1\right)^2}\right\} \\
&= \mathcal{L}^{-1}\left\{\mathcal{L}\left\{\sin t\right\}\mathcal{L}\left\{\frac{\sin t}{2} - \frac{t\cos t}{2}\right\}\right\} \\
&= \sin t * \left(\frac{\sin t}{2} - \frac{t \cos t}{2}\right) \\
&= \int_0^t \sin\left(t-u\right)\left(\frac{\sin u}{2} - \frac{u\cos u}{2}\right) du \\
&= \frac{1}{2}\int_0^t \sin\left(t-u\right)\sin u du - \frac{1}{2}\int_0^t u \cos u \sin t\left(t-u\right) du \\
&= \frac{1}{2}\int_0^t \left(\sin t \cos u \sin u - \cos t \sin^2 u\right) du - \frac{1}{2}\int_0^t \left(u\cos^2 u \sin t - u \cos t \cos u \sin u\right) du
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\sin^3 t}{4} - \frac{\cos t}{4}\int_0^t (1-\cos 2u)\,du - \frac{1}{2}\int_0^t \big(u\cos^2 u\sin t - u\cos t\cos u\sin u\big)\,du \\
&= \frac{\sin^3 t}{4} - \frac{t\cos t}{4} + \frac{\sin t\cos^2 t}{2} - \frac{1}{2}\int_0^t \big(u\cos^2 u\sin t - u\cos t\cos u\sin u\big)\,du \\
&= \frac{\sin t\left(1-\cos^2 t\right)}{4} - \frac{t\cos t}{4} + \frac{\sin t\cos^2 t}{2} - \frac{1}{2}\int_0^t \big(u\cos^2 u\sin t - u\cos t\cos u\sin u\big)\,du \\
&= \frac{\sin t}{4} - \frac{t\cos t}{4} - \frac{\sin t}{2}\int_0^t u\cos^2 u du + \frac{\cos t}{2}\int_0^t u\cos u\sin u du \\
&= \frac{\sin t}{4} - \frac{t\cos t}{4} - \frac{\sin t}{4}\int_0^t u\,(1+\cos 2u)\,du + \frac{\cos t}{2}\int_0^t u\cos u\sin u du
\end{aligned}
$$

mas

$$
\begin{aligned}
&-\frac{\sin t}{4}\int_0^t u\,(1+\cos 2u)\,du + \frac{\cos t}{2}\int_0^t u\cos u\sin u du \\
&= -\frac{\sin t}{4}\left(u\left(u+\frac{\sin 2u}{2}\right)\Big|_0^t - \int_0^t\left(u+\frac{\sin 2u}{2}\right)du\right) + \frac{\cos t}{2}\left(\frac{u\sin^2 u}{2}\Big|_0^t - \frac{1}{2}\int_0^t \sin^2 u du\right) \\
&= -\frac{\sin t}{4}\left(t\left(t+\frac{\sin 2t}{2}\right) - \left[\frac{u^2}{2} - \frac{\cos 2u}{4}\right]_0^t\right) + \frac{\cos t}{2}\left(\frac{t\sin^2 t}{2} - \frac{1}{4}\int_0^t (1-\cos 2u)\,du\right) \\
&= -\frac{\sin t}{4}\left(t^2 + \frac{t\sin 2t}{2} - \frac{t^2}{2} + \frac{\cos 2t}{4} - \frac{1}{4}\right) + \frac{\cos t}{2}\left(\frac{t\sin^2 t}{2} - \frac{1}{4}\left[u - \frac{\sin 2u}{2}\right]_0^t\right) \\
&= -\frac{\sin t}{4}\left(\frac{t^2}{2} + \frac{t\sin 2t}{2} + \frac{\cos 2t}{4} - \frac{1}{4}\right) + \frac{\cos t}{2}\left(\frac{t\sin^2 t}{2} - \frac{t}{4} + \frac{\sin 2t}{8}\right) \\
&= -\frac{\sin t}{4}\left(\frac{t^2}{2} + t\sin t\cos t + \frac{\cos 2t - 1}{4}\right) + \frac{t\cos t\sin^2 t}{4} - \frac{t\cos t}{8} + \frac{\sin t\cos^2 t}{8} \\
&= -\frac{\sin t}{4}\left(\frac{t^2}{2} + t\sin t\cos t - \frac{\sin^2 t}{2}\right) + \frac{t\cos t\sin^2 t}{4} - \frac{t\cos t}{8} + \frac{\sin t\cos^2 t}{8} \\
&= -\frac{t^2\sin t}{8} - \frac{t\sin^2 t\cos t}{4} + \frac{\sin^3 t}{8} + \frac{t\cos t\sin^2 t}{4} - \frac{t\cos t}{8} + \frac{\sin t\cos^2 t}{8} \\
&= -\frac{t^2\sin t}{8} + \frac{\sin^3 t}{8} - \frac{t\cos t}{8} + \frac{\sin t\cos^2 t}{8} \\
&= -\frac{t^2\sin t}{8} - \frac{t\cos t}{8} + \frac{\sin t}{8}
\end{aligned}
$$

e portanto

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{\left(s^2+1\right)^3}\right\} &= \frac{\sin t}{4} - \frac{t\cos t}{4} - \frac{t^2\sin t}{8} - \frac{t\cos t}{8} + \frac{\sin t}{8} \\
&= -\frac{t^2\sin t}{8} - \frac{3}{8}t\cos t\frac{3}{8}\sin t
\end{aligned}
$$

- $\frac{s^2}{\left(s^2+1\right)^4}$

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s^2}{\left(s^2+1\right)^4}\right\} &= \mathcal{L}^{-1}\left\{\frac{s}{\left(s^2+1\right)^2}\frac{s}{\left(s^2+1\right)^2}\right\} \\
&= \mathcal{L}^{-1}\left\{\mathcal{L}\left\{\frac{t\sin t}{2}\right\}\mathcal{L}\left\{\frac{t\sin t}{2}\right\}\right\} \quad \text{por e)} \\
&= \frac{1}{4}\mathcal{L}^{-1}\left\{\mathcal{L}\left\{t\sin t\right\}\mathcal{L}\left\{t\sin t\right\}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4}(t\sin t) * (t\sin t) \\
&= \frac{1}{4}\int_0^t (t-u)\sin(t-u)\, u\sin u du \\
&= \frac{1}{4}\int_0^t (t-u)\left(\sin t\cos u\sin u - \cos t\sin^2 u\right) du \\
&= \ldots \\
&= \frac{t\sin t - t^2\cos t}{16}
\end{aligned}
$$

4.6. **Sistemas de Equações Diferenciais lineares**

Exercício 4.6.1.

a) $y' = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} y + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t, \quad y(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$
y' = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} y + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t \Leftrightarrow \begin{cases} y_1' = y_1 + 4y_2 + e^t \\ y_2' = y_1 + y_2 + e^t \end{cases}
$$

Aplicando transformadas de Laplace ($Y_1 = \mathcal{L}\{y_1\}$ e $Y_2 = \mathcal{L}\{y_2\}$) obtemos o seguinte sistema

$$
\begin{cases} sY_1 - 2 = Y_1 + 4Y_2 + \frac{1}{s-1} \\ sY_2 - 1 = Y_1 + Y_2 + \frac{1}{s-1} \end{cases}
\Leftrightarrow \begin{cases} (s-1)Y_1 - 4Y_2 = \frac{1}{s-1} + 2 \\ -Y_1 + (s-1)Y_2 = \frac{1}{s-1} + 1 \end{cases}
$$

$$
\Leftrightarrow \begin{cases} (s-1)Y_1 - 4Y_2 = \frac{2s-1}{s-1} \\ -Y_1 + (s-1)Y_2 = \frac{s}{s-1} \end{cases}
$$

$$
\Leftrightarrow \begin{cases} Y_1 = \frac{1}{4(s+1)} - \frac{1}{s-1} + \frac{11}{4(s-3)} \\ Y_2 = -\frac{1}{8(s+1)} - \frac{1}{4(s-1)} + \frac{11}{8(s-3)} \end{cases}
$$

$$
\Leftrightarrow \begin{cases} y_1 = \mathcal{L}^{-1}\left\{\frac{1}{4(s+1)} - \frac{1}{s-1} + \frac{11}{4(s-3)}\right\} \\ y_2 = \mathcal{L}^{-1}\left\{-\frac{1}{8(s+1)} - \frac{1}{4(s-1)} + \frac{11}{8(s-3)}\right\} \end{cases}
$$

$$
\Leftrightarrow \begin{cases} y_1 = \frac{1}{4}e^{-t} - e^t + \frac{11}{4}e^{3t} \\ y_2 = -\frac{1}{8}e^{-t} - \frac{1}{4}e^t + \frac{11}{8}e^{3t} \end{cases}
$$

b) $y' = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix} y, \quad y(0) = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$

$$
y' = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix} y \Leftrightarrow \begin{cases} y_1' = y_1 - 3y_2 \\ y_2' = -2y_1 + 2y_2 \end{cases}
$$

Aplicando transformadas de Laplace ($Y_1 = \mathcal{L}\{y_1\}$ e $Y_2 = \mathcal{L}\{y_2\}$) obtemos o seguinte sistema

$$
\begin{cases} sY_1 = Y_1 - 3Y_2 \\ sY_2 - 5 = -2Y_1 + 2Y_2 \end{cases}
\Leftrightarrow \begin{cases} (s-1)Y_1 + 3Y_2 = 0 \\ 2Y_1 + (s-2)Y_2 = 5 \end{cases}
$$

$$
\Leftrightarrow \begin{cases} Y_1 = \frac{3}{s+1} - \frac{3}{s-4} \\ Y_2 = \frac{2}{s+1} + \frac{3}{s-4} \end{cases}
$$

$$
\Leftrightarrow \begin{cases} y_1 = \mathcal{L}^{-1}\left\{\frac{3}{s+1} - \frac{3}{s-4}\right\} \\ y_2 = \mathcal{L}^{-1}\left\{\frac{2}{s+1} + \frac{3}{s-4}\right\} \end{cases}
$$

$$
\Leftrightarrow \begin{cases} y_1 = 3e^{-t} - 2e^{4t} \\ y_2 = 2e^{-t} + 3e^{4t} \end{cases}
$$

c) $y' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} y + \begin{pmatrix} t \\ 3e^t \end{pmatrix}, \quad y(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$y' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} y + \begin{pmatrix} t \\ 3e^t \end{pmatrix} \Leftrightarrow \left\{ \begin{array}{c} y_1' = 3y_1 - 2y_2 + t \\ y_2' = 2y_1 - 2y_2 + 3e^t \end{array} \right.$$

Aplicando transformadas de Laplace ($Y_1 = \mathcal{L}\{y_1\}$ e $Y_2 = \mathcal{L}\{y_2\}$) obtemos o seguinte sistema

$$\left\{ \begin{array}{lcl} sY_1 - 2 & = & 3Y_1 - 2Y_2 + \frac{1}{s} \\ sY_2 - 1 & = & 2Y_1 - 2Y_2 + \frac{3}{s-1} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{lcl} (s-3)\,Y_1 + 2Y_2 & = & \frac{1}{s} + 2 \\ -2Y_1 + (s+2)\,Y_2 & = & \frac{2+s}{s-1} \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{lcl} Y_1 & = & -\frac{1}{s} - \frac{2}{3(s+1)} + \frac{3}{s-1} + \frac{2}{3(s-2)} \\ Y_2 & = & -\frac{1}{s} - \frac{4}{3(s+1)} + \frac{3}{s-1} + \frac{1}{3(s-2)} \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{lcl} \mathcal{L}^{-1}\{Y_1\} & = & \mathcal{L}^{-1}\left\{ -\frac{1}{s} - \frac{2}{3(s+1)} + \frac{3}{s-1} + \frac{2}{3(s-2)} \right\} \\ \mathcal{L}^{-1}\{Y_2\} & = & \mathcal{L}^{-1}\left\{ -\frac{1}{s} - \frac{4}{3(s+1)} + \frac{3}{s-1} + \frac{1}{3(s-2)} \right\} \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{lcl} y_1 & = & -1 - \frac{2}{3}e^{-t} + 3e^t + \frac{2}{3}e^{2t} \\ y_2 & = & -1 - \frac{4}{3}e^{-t} + 3e^t + \frac{e^{2t}}{2} \end{array} \right.$$

d) $y' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} y + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t, \quad y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$y' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} y + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t \Leftrightarrow \left\{ \begin{array}{c} y_1' = 3y_1 - 4y_2 + e^t \\ y_2' = y_1 - y_2 + e^t \end{array} \right.$$

Aplicando transformadas de Laplace ($Y_1 = \mathcal{L}\{y_1\}$ e $Y_2 = \mathcal{L}\{y_2\}$) obtemos o seguinte sistema

$$\left\{ \begin{array}{lcl} sY_1 - 1 & = & 3Y_1 - 4Y_2 + \frac{1}{s-1} \\ sY_2 - 1 & = & Y_1 - Y_2 + \frac{1}{s-1} \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{lcl} (s-3)\,Y_1 + 4Y_2 & = & \frac{s}{s-1} \\ -Y_1 + (s+1)\,Y_2 & = & \frac{s}{s-1} \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{lcl} Y_1 & = & -\frac{3}{4(s-1)} + \frac{1}{s^2-2s-7}\left(\frac{7}{4}s + \frac{21}{4}\right) \\ Y_2 & = & \frac{1}{8(s-1)} + \frac{1}{s^2-2s-7}\left(\frac{7}{8}s - \frac{7}{8}\right) \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{lcl} \mathcal{L}^{-1}\{Y_1\} & = & \mathcal{L}^{-1}\left\{ -\frac{3}{4(s-1)} + \frac{\frac{7}{4}s+\frac{21}{4}}{s^2-2s-7} \right\} \\ \mathcal{L}^{-1}\{Y_2\} & = & \mathcal{L}^{-1}\left\{ \frac{1}{8(s-1)} + \frac{\frac{7}{8}s-\frac{7}{8}}{s^2-2s-7} \right\} \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{lcl} y_1 & = & -\frac{3}{4}e^t + \frac{7}{4}\mathcal{L}^{-1}\left\{ \frac{s-1+4}{(s-1)^2-8} \right\} \\ y_2 & = & \frac{e^t}{8} + \frac{7}{8}\mathcal{L}^{-1}\left\{ \frac{s-1}{(s-1)^2-8} \right\} \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{lcl} y_1 & = & -\frac{3}{4}e^t + \frac{7}{4}\left(\mathcal{L}^{-1}\left\{ \frac{s-1}{(s-1)^2-\left(2\sqrt{2}\right)^2} \right\} + \sqrt{2}\mathcal{L}^{-1}\left\{ \frac{2\sqrt{2}}{(s-1)^2-\left(2\sqrt{2}\right)^2} \right\} \right) \\ y_2 & = & \frac{e^t}{8} + \frac{7}{8}\mathcal{L}^{-1}\left\{ \frac{s-1}{(s-1)^2-\left(2\sqrt{2}\right)^2} \right\} \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{lcl} y_1 & = & -\frac{3}{4}e^t + \frac{7}{4}\left(e^t\cosh\left(2\sqrt{2}t\right) + \sqrt{2}e^t\sinh\left(2\sqrt{2}t\right)\right) \\ y_2 & = & \frac{e^t}{8} + \frac{7}{8}e^t\cosh\left(2\sqrt{2}t\right) \end{array} \right.$$

[1] Ilda Marisa de Sá Reis – Departamento de Matemática 2005/06 – ESTiG - IPB