Transformada de Laplace Directa ———————————————

Exercício 1. Calcular as transformadas de Laplace das funções.

$(a)\ 2$ $(b)\ e^{-2t}$ $(c)\ \frac{t}{2}$ $(d)\ t^2$

$(e)\ t^2e^{-t}$ $(f)\ e^{-5t}sen(8t)$ $(g)\ t^3cos(6t)$ $(h)\ sen(t+\pi/4)$

$(i)\ u(t)$ $(j)\ u(t-0.3)$ $(k)\ 5\delta(t-2)$ $(l)\ \delta(t-5).t^2$

$(m)\ u(t-1)sin(t-1)$ $(n)\ u(t-1)sin(t)$ $(o)\ u(t-0.3)(t-1)^2$

$(p)\ u(t-2)t^2$

1. Resolução

(a) $\mathscr{L}(2) = 2\mathscr{L}(1) \underset{\text{F. 1}}{=} \frac{2}{s}$

(b) $\mathscr{L}\left(e^{-2t}\right) \underset{\text{F. 15; } a=-2}{=} \frac{1}{s+2}$

(c) $\mathscr{L}\left(\frac{t}{2}\right) = \frac{1}{2}\mathscr{L}(t) \underset{\text{F. 11; } n=1}{=} \frac{1}{2s^2}$

(d) $\mathscr{L}\left(t^2\right) \underset{\text{F. 11; } n=2}{=} \frac{2}{s^3}$

(e) $\mathscr{L}\left(t^2e^{-t}\right) \underset{\text{F. 7; } n=2}{=} (-1)^2\frac{d^2}{ds^2}\mathscr{L}\left(e^{-t}\right) \underset{\text{F. 15; } a=-1}{=} (-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s+1}\right) = \frac{2}{(s+1)^3}$

(f) $\mathscr{L}\left(e^{-5t}sen(8t)\right) \underset{\text{F. 22; } a=-5; k=8}{=} \frac{8}{(s+5)^2+64}$

(g) $\mathscr{L}\left(t^3cos(6t)\right) \underset{\text{F. 7; } n=3}{=} (-1)^3\frac{d^3}{ds^3}\mathscr{L}\left(cos(6t)\right) \underset{\text{F. 14; } k=6}{=} -\frac{d^3}{ds^3}\left(\frac{s}{s^2+36}\right)$

$= -\frac{6(s^4-216s^2+1296)}{(s^2+36)^4}$

(h) $\mathscr{L}\left(sen(t+\pi/4)\right) = \mathscr{L}\left(\frac{\sqrt{2}}{2}sen(t)+\frac{\sqrt{2}}{2}cos(t)\right)$

obs. $sen(a+b) = sen(a)cos(b)+sen(b)cos(a)$

$$\mathscr{L}\left(\frac{\sqrt{2}}{2}sen(t)+\frac{\sqrt{2}}{2}cos(t)\right) = \mathscr{L}\left(\frac{\sqrt{2}}{2}sen(t)\right)+\mathscr{L}\left(\frac{\sqrt{2}}{2}cos(t)\right)$$

$$= \frac{\sqrt{2}}{2}\mathscr{L}\left(sen(t)\right)+\frac{\sqrt{2}}{2}\mathscr{L}\left(cos(t)\right) \underset{\text{F. s 13, 14}}{=} \frac{\sqrt{2}}{2}\frac{1}{s^2+1}+\frac{\sqrt{2}}{2}\frac{s}{s^2+1}$$

(i) $\mathscr{L}\left(u(t)\right) \underset{\text{F. 3, } a=0}{=} \frac{1}{s}$

(j) $\mathscr{L}\left(u(t-0.3)\right) \underset{\text{F. 3, } a=0.3}{=} \frac{e^{-0.3s}}{s}$

(k) $\mathscr{L}\left(5\delta(t-2)\right) \underset{\text{F. 6}}{=} 5e^{-2s}$

(l) $\mathscr{L}\left(\delta(t-5)t^2\right) \underset{\text{F. 38, } a=5}{=} 5^2e^{-5s} = 25e^{-5s}$

A resolução das alíneas seguintes aplica o teorema da translação em t:

$$u(t-a)f(t-a) = e^{-as}F(s)$$

(m) $u(t-1)f(t-1) = u(t-1)sin(t-1) \Rightarrow f(t) = sin(t)$

$F(s) = \mathscr{L}(sin(t)) \underset{\text{F. 13; k=1}}{=} \frac{1}{s^2+1}$

$\mathscr{L}(u(t-1)sin(t-1)) \underset{\text{F. 4; a=1}}{=} \frac{e^{-s}}{s^2+1}$

(n) $u(t-1)f(t-1) = u(t-1)sin(t) \Rightarrow f(t-1) = sin(t) \Rightarrow f(t) = f((t+1)-1) = sin(t+1)$

Como $sin(t+1) = sin(t)sin(1) + cos(t)sin(1)$, temos

$F(s) = \mathscr{L}(sin(t)cos(1) + cos(t)sin(1)) = cos(1)\mathscr{L}(sin(t)) + sin(1)\mathscr{L}(cos(t)$

$\underset{\text{F. 13, 14; k=1}}{=} \frac{cos(1)}{s^2+1} + \frac{s\ sin(1)}{s^2+1}$

$\mathscr{L}(u(t-1)sin(t)) \underset{\text{F. 4; a=1}}{=} e^{-s}\left(\frac{cos(1)}{s^2+1} + \frac{s\ sin(1)}{s^2+1}\right)$

(o) $u(t-0.3)f(t-0.3) = u(t-0.3)(t-1)^2 \Rightarrow f(t-0.3) = (t-1)^2 \Longrightarrow f(t) = f((t+0.3)-0.3) = (t-0.7)^2$

$F(s) = \mathscr{L}((t-0.7)^2) = \mathscr{L}(t^2 - 1.4t + 0.49) = \mathscr{L}(t^2) - 1.4\mathscr{L}(t) + \mathscr{L}(0.49)$

$\underset{\text{F. 11, 1}}{=} \frac{2}{s^3} + \frac{-1.4}{s^2} + \frac{0.49}{s}$

$\mathscr{L}(u(t-0.3)(t-1)^2 \underset{\text{F. 4; a=0.3}}{=} e^{-0.3s}\left(\frac{2}{s^3} + \frac{-1.4}{s^2} + \frac{0.49}{s}\right)$

(p) $u(t-2)f(t-2) = u(t-2)t^2 \Rightarrow f(t-2) = t^2 \Rightarrow f(t) = f((t+2)-2) = (t+2)^2$

$F(s) = \mathscr{L}((t+2)^2) = \mathscr{L}(t^2+2t+4) = \mathscr{L}(t^2) + 2\mathscr{L}(t) + \mathscr{L}(4)$

$\underset{\text{F. 11, 1}}{=} \frac{2}{s^3} + \frac{2}{s^2} + \frac{4}{s}$

$\mathscr{L}(u(t-2)t^2)) \underset{\text{F. 4; a=2}}{=} e^{-2s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{4}{s}\right)$

Exercício 2. Calcular as transformadas de Laplace das funções (assumir $x(0) = 1,\ \dot{x}(0) = 2,\ \ddot{x}(0) = 3$).

$(a)\ f(t) = a + bt + ct^2$ $\quad (b)\ f(t) = 2e^{at} - e^{-at}$ $\quad (c)\ f(t) = 3\ddot{x} + 4\dot{x} - 2x$

$(d)\ f(t) = 2\dot{x} - x + 2$ $\quad (e)\ f(t) = \dddot{x} + \ddot{x} - 2e^{-t}$ $\quad (f)\ f(t) = x^{(4)}(t)$

2. Resolução

(a) $\mathscr{L}\left(a + bt + ct^2\right) = \mathscr{L}(a) + b\mathscr{L}(t) + b\mathscr{L}\left(t^2\right) \underset{\text{F. 11}}{=} \frac{a}{s} + \frac{b}{s^2} + \frac{2c}{s^3} +$

(b) $\mathscr{L}\left(2e^{at} - e^{-at}\right) = 2\mathscr{L}\left(e^{at}\right) - \mathscr{L}\left(e^{-at}\right) \underset{\text{F. 15}}{=} \frac{2}{s-a} - \frac{1}{s+a}$

(c) $= \mathscr{L}(3\ddot{x} + 4\dot{x} - 2x) = 3\mathscr{L}(\ddot{x}) + 4\mathscr{L}(\dot{x}) - 2\mathscr{L}(x) = 3(s^2X(s) - s - 2) + 4(sX(s) - 1) - 2X(s)$

$= X(s)(3s^2 + 4s - 2) - 3s - 10$

(d) $\mathscr{L}(2\dot{x} - x + 2) = 2\mathscr{L}(\dot{x}) - \mathscr{L}(x) + 2\mathscr{L}(1) = 2(sX(s) - x(0)) - X(s) + 2/s$

$= X(s)(2s-1) + 2/s - 2$

(e) $\mathscr{L}\left(\dddot{x} + \ddot{x} - 2e^{-t}\right) = \mathscr{L}(\dddot{x}) + \mathscr{L}(\ddot{x}) - 2\mathscr{L}\left(e^{-t}\right)$

$= (s^3X(s) - s^2 - 2s - 3) + (s^2X(s) - s - 2) - \frac{2}{s+1} = X(s)(s^3 + s^2) - s^2 - 3s - 5 - \frac{2}{s+1}$

(f) $\mathscr{L}\left(x^{(4)}(t)\right) = s^4X(s) - s^3x(0) - s^2x'(0) - sx''(0) - x'''(0)$

Transformada Inversa de Laplace ———————————

Exercício 3. Calcular as transformadas inversas de Laplace das funções.

$(a)\ F(s) = \dfrac{1}{s}$ $\quad (b)\ F(s) = \dfrac{3}{s^2+2}$ $\quad (c)\ F(s) = \dfrac{1}{s-8}$ $\quad (d)\ F(s) = \dfrac{3}{s^2-2}$

$(e)\ F(s) = \dfrac{1}{s+8}$ $\quad (f)\ F(s) = \dfrac{1}{s^2}$ $\quad (g)\ F(s) = \dfrac{1}{(s+1)^2}$ $\quad (h)\ F(s) = \dfrac{6}{s^4}$

3. Resolução

$$\textbf{(a)}\ \mathscr{L}^{-1}\left(\frac{3}{s}\right) = 3\mathscr{L}^{-1}\left(\frac{1}{s}\right) \underset{\text{F. 1}}{=} 3$$

$$\textbf{(b)}\ \mathscr{L}^{-1}\left(\frac{3}{s^2+2}\right) = \frac{3}{\sqrt{2}}\mathscr{L}^{-1}\left(\frac{\sqrt{2}}{s^2+2}\right) \underset{\text{F. 13, } k=\sqrt{2}}{=} \frac{3}{\sqrt{2}} sen(\sqrt{2}t)$$

$$\textbf{(c)}\ \mathscr{L}^{-1}\left(\frac{1}{s-8}\right) \underset{\text{F. 15, } a=8}{=} e^{8t}$$

$$\textbf{(d)}\ \mathscr{L}^{-1}\left(\frac{3}{s^2-2}\right) = \frac{3}{\sqrt{2}}\mathscr{L}^{-1}\left(\frac{\sqrt{2}}{s^2-2}\right) \underset{\text{F. 16, } k=\sqrt{2}}{=} \frac{3}{\sqrt{2}} senh(\sqrt{2}t)$$

$$\textbf{(e)}\ \mathscr{L}^{-1}\left(\frac{1}{s+8}\right) \underset{\text{F. 15, } a=-8}{=} e^{-8t}$$

$$\textbf{(f)}\ \mathscr{L}^{-1}\left(\frac{1}{s^2}\right) \underset{\text{F. 11, } n=1}{=} t$$

$$\textbf{(g)}\ \mathscr{L}^{-1}\left(\frac{1}{(s+2)^2}\right) \underset{\text{F. 20, } a=-2}{=} te^{-2t}$$

$$\textbf{(h)}\ \mathscr{L}^{-1}\left(\frac{6}{s^4}\right) \underset{\text{F. 11, } n=3}{=} t^3$$

Exercício 4. Calcular as transformadas inversas de Laplace das funções.

$(a)\ F(s) = \dfrac{1}{s^2+6s+3}$ $\quad (b)\ F(s) = \dfrac{s}{s^2+s+1}$ $\quad (c)\ F(s) = \dfrac{6}{s(s+1)}$

$(d)\ F(s) = \dfrac{4}{s(s^2+2)}$ $\quad (e)\ F(s) = \dfrac{6}{(s+1)^2+4}$ $\quad (f)\ F(s) = \dfrac{3}{s(s+1)^2(s-3)}$

$(g)\ F(s) = \dfrac{2}{s-1} - \dfrac{3}{s^2+5}$ $\quad (h)\ F(s) = e^{-3s}\dfrac{6}{s(s+1)}$ $\quad (i)\ F(s) = arctan\left(\dfrac{6}{s}\right)$

4. Resolução

(a) $\mathscr{L}^{-1}\left(\frac{1}{s^2+6s+3}\right) = \mathscr{L}^{-1}\left(\frac{1}{(s+3)^2-6}\right) = \frac{1}{\sqrt{6}}\mathscr{L}^{-1}\left(\frac{1}{\sqrt{6}}\frac{\sqrt{6}}{(s+3)^2-6}\right)$

$\underset{\text{F. 24 } k=\sqrt{6},\ a=-3}{=} \frac{1}{\sqrt{6}}e^{-3t}senh(\sqrt{6}t)$

(b) $\mathscr{L}^{-1}\left(\frac{1}{s^2+s+1}\right) = \mathscr{L}^{-1}\left(\frac{1}{(s+\frac{1}{2})^2+\frac{3}{4}}\right) = \frac{2}{\sqrt{3}}\mathscr{L}^{-1}\left(\frac{\sqrt{3}/2}{(s+\frac{1}{2})^2+\frac{3}{4}}\right)$

$\underset{\text{F. 22 } k=\sqrt{3}/2,\ a=-1/2}{=} \frac{2}{\sqrt{3}}e^{-t/2}sen(\sqrt{3}t/2)$

(c) $\mathscr{L}^{-1}\left(\frac{6}{s(s+1)}\right) = 6\mathscr{L}^{-1}\left(\frac{1}{s(s+1)}\right) \underset{\text{F. 18, } a=0,\ b=-1}{=} 6(1-e^{-t})$

(d) $\mathscr{L}^{-1}\left(\frac{4}{s(s^2+4)}\right)$

obs. $\frac{4}{s(s^2+4)} = \frac{1}{s} + \frac{-s}{s^2+4}$

$\mathscr{L}^{-1}\left(\frac{4}{s(s^2+4)}\right) = \mathscr{L}^{-1}\left(\frac{1}{s}\right) + \mathscr{L}^{-1}\left(\frac{-s}{s^2+4}\right) \underset{\text{F. s 1, 14}}{=} 1-cos(2t)$

(e) $\mathscr{L}^{-1}\left(\frac{6}{(s+1)^2+4}\right) = 3\mathscr{L}^{-1}\left(\frac{2}{(s+1)^2+4}\right) \underset{\text{F. 22, } a=-1,\ k=2}{=} 3e^{-t}sen(2t)$

(f) $\mathscr{L}^{-1}\left(\frac{3}{s(s+1)^2(s-3)}\right)$

obs. $\frac{3}{s(s+1)^2(s-3)} = \frac{-1}{s} + \frac{3/4}{(s+1)^2} + \frac{25/16}{s+1} + \frac{1/16}{s-3}$

$\mathscr{L}^{-1}\left(\frac{3}{s(s+1)^2(s-3)}\right) = \mathscr{L}^{-1}\left(\frac{-1}{s}\right) + \mathscr{L}^{-1}\left(\frac{3/4}{(s+1)^2}\right) + \mathscr{L}^{-1}\left(\frac{25/16}{s+1}\right) + \mathscr{L}^{-1}\left(\frac{1/16}{s-3}\right)$

$= -1 + \frac{3}{4}te^{-t} + \frac{25}{16}e^{-t} + \frac{1}{16}e^{3t}$

(g) $\mathscr{L}^{-1}\left(\frac{2}{s-1} - \frac{3}{s^2+5}\right) = 2e^{t} - \frac{3}{\sqrt{5}}sen(\sqrt{5}t)$

(h) $\mathscr{L}^{-1}\left(e^{-3s}\frac{6}{s(s+1)}\right) \underset{\text{F. 4}}{=} u(t-3)6\left(1-e^{-(t-3)}\right)$

(i) $\mathscr{L}^{-1}\left(arctan(6/s)\right) = sen(6t)/t$